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A Short Note on Cointegration and Aggregation

by

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1. In Lippi (1988) two cases are considered in which micro cointegration implies macro cointegration:

(a) equal cointegrating vectors at the micro level;

(b) equal (up to a constant term) right-hand variables across the microequations, i.e. $x_{it} = \theta_i X_t$, where X_t is $I(1)$ and $\theta_i \neq 0.$*

Gonzalo (1989) pointed out another case in which micro cointegration implies macro cointegration, namely:

(c) $x_{it} = \theta_i X_t + I(0)$, where X_t is $I(1)$ and $\theta_i \neq 0.$**

Cases (a), (b), (c), all have a clear economic meaning. Case (c) generalizes (b) and is therefore much more interesting if correspondence to empirical situations is under discussion.

Here the cointegration-aggregation issue is examined by means of an elementary geometric representation. It is argued that (a) and (c) are not only sufficient for micro to imply macro cointegration, but also that, if cases having no economic interest are excluded, such implication occurs only if either (a) or (c) occur (this result is slightly more complicated, but only as regards (a), if the aggregation is done by weighted sums instead of simple sums). Lastly, some considerations on the occurrence of (c) in empirical cases are put forward. Inciden-

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* See Lippi (1988), p. 570 and section 5.1 (not section 4, as misprinted on p. 570).

** Gonzalo's paper deals with a more general model, in which the microequations contain k variables on the right hand side, with $k \geq 1$. Here I shall limit myself to the special case when $k = 1$.

tally, the non-linear case of logarithmic aggregation will be briefly, and rather informally, discussed.

As micro cointegration will be assumed throughout the paper, I shall simply speak of macro cointegration (occurring or not), avoiding the expression "micro implies (or does not) macro cointegration".

2. Let us begin by restating the following elementary proposition (Lippi (1988), p. 570):

Fact 1. Assume that:

$$x(t) = (\; x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)\;)$$

is $I(1)$ and that $X(t) = \sum \mu_i x_i(t)$ is $I(1)$ as well; moreover, assume that:

$$y(t) = (\; y_1(t) \quad y_2(t) \quad \cdots \quad y_n(t)\;)$$

is $I(1)$ and that $Y(t) = \sum \lambda_i y_i(t)$ is $I(1)$ (in other words, the aggregating vectors λ and μ are not cointegrating, respectively, for $x(t)$ and $y(t)$). Lastly assume that $(y_i(t) \; x_i(t))$ is cointegrated by $(1 \; - c_i)$, $c_i \neq 0$, for $i = 1, 2, \ldots, n$. $Y(t)$ and $X(t)$ are cointegrated if and only if one of the following conditions holds:
(1) There exists a real $c \neq 0$ such that $\lambda_i c_i - \mu_i c = 0$ for any i.
(2) There exists a real $c \neq 0$ such that the vector whose components are $\lambda_i c_i - \mu_i c$ is cointegrating for $x(t)$.

The proof of Fact 1 is trivially obtained by expanding the cointegration condition:

$$Y(t) - cX(t) = I(0).$$

The assumptions made in Fact 1 will be kept throughout the paper.

Now consider the cointegrating subspace of $x(t)$, i.e. the subspace of $\mathbf{R}^n$ containing all the cointegrating vectors of $x(t)$ plus the null vector: call it V_x (if $x(t)$ is not cointegrated V_x is the null subspace).

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The dimension of V_x is the cointegration rank of $x(t)$. Let us consider the straight line $S = P + sQ$, where

$$P = (\lambda_1 c_1 \ \lambda_2 c_2 \ \cdots \lambda_n c_n), \quad Q = (-\mu_1 \ -\mu_2 \ \cdots - \mu_n),$$

while s is a real parameter. Since P is not an element of V_x (otherwise $Y(t) = \sum \lambda_i y_i(t) = \sum \lambda_i c_i x_i(t) + I(0)$ would be $I(0)$), the intersection $S \cap V_x$ can contain no more than one point (vector). If $S \cap V_x$ is empty then $Y(t)$ and $X(t)$ are not cointegrated. If it is not empty and the only vector it contains is null then condition (1) holds, if such a vector is not null then condition (2) holds. Notice that conditions (1) and (2) are mutually exclusive*, but notice also that condition (1) does not exclude cointegration for $x(t)$, it only excludes the possibility of vectors in S being cointegrating for $x(t)$.

In the Figure below conditions (1) and (2) are represented for $n = 2$ with $x(t)$ cointegrated, so that $\dim(V_x)=1$. Notice that in this case in order to have an empty intersection between S and V_x, $x(t)$ should be non cointegrated.

Condition (2) Condition (1)

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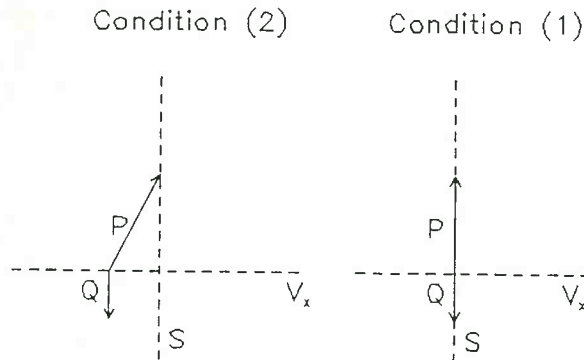
* This is not completely trivial: conditions (1) and (2) could hold with different c's. But in this case the line S would be entirely contained in V_x.

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In general since Q is not contained in V_x (otherwise μ would be cointegrating for $x(t)$, i.e. $X(t)$ would be $I(0)$) the line S can not be parallel to the subspace V_x. Therefore, if the cointegration rank of $x(t)$ is $n-1$, then $S \cap V_x$ can not be empty, so that either condition (1) or condition (2) must hold:

Fact 2. If the cointegration rank of x_t is $n-1$, i.e. $\dim(V_x) = n-1$, then $Y(t)$ and $X(t)$ are cointegrated.

Lastly, let us consider this further case:

(d) $x(t)$ is $I(1)$, $x_i(t) = \alpha_{ij} x_j(t) + I(0)$, with $\alpha_{ij} \neq 0$ for any i and j.
   We have:

Fact 3. Let $x(t)$ be $I(1)$. Case (c), case (d), and $\dim(V_x) = n-1$ are equivalent.

Case (c) and (d) are trivially equivalent. In the same way, case (c) trivially implies that $\dim(V_x) = n-1$. If $\dim(V_x) = n-1$, the hyperplane V_x can not contain any one of the vectors of the natural basis of $\mathbf{R}^n$ (otherwise one of the components of $x(t)$ would be stationary). (d) follows from elementary geometric considerations: for instance, the plane through $(1\ 0\ \cdots\ 0)$ and $(0\ 1\ \cdots\ 0)$ must intersect V_x in vectors $(a\ b\ \cdots\ 0)$ with $a \neq 0$, $b \neq 0$.

3. When $\lambda_i / \mu_i = \lambda_j / \mu_j$, for any i and j (in particular $\lambda_i = \mu_i = 1$), condition (1) is equivalent to $c_i = c_j$, for any i and j, i.e. equivalent to case (a). But if $\lambda_i / \mu_i \neq \lambda_j / \mu_j$, for some i and j, the equality of the micro cointegrating vectors implies that condition (1) does not hold, so that the aggregates are cointegrated only if condition (2) holds. Let us consider the following example. Suppose $n = 2$ and:

$$\lambda = (.5\ .5)$$
$$\mu = (.3\ .7)$$
$$c_1 = c_2 = 1.$$

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Moreover, assume that $x_2(t) = kx_1(t) + I(0)$. Then:

$$Y(t) - cX(t) = .5y_1(t) + .5y_2(t) - (.3x_1(t) + .7x_2(t))$$
$$= (.5 - .3c)x_1(t) + (.5 - .7c)x_2(t) + I(0)$$
$$= [.5(1 + k) - c(.3 + .7k)]x_1(t) + I(0),$$

so that:

$$c = \frac{.5(1 + k)}{.3 + .7k}.$$

Here condition (1) is not fulfilled but cointegration of $x(t)$ ensures cointegration of the aggregates. Notice that $c_1 = c_2$ does not imply $c = c_1 = c_2$. This occurs only when $k = 1$.*

In Lippi (1988), section 3.4, the case of logarithmic aggregation is considered, i.e. the case when the macrovariables are the logarithms of the simple sums of the microvariables. Under the assumptions that

$$\log(y_i(t)) = c\log(x_i(t)) + I(0) \tag{†}$$
$$\log(x_i(t)) = \log(x_j(t)) + I(0), \tag{‡}$$

for any i and j, and that the variances of the $I(0)$ variables in (†) and (‡) are small, it is shown that the aggregate variables (i.e. the macro-logs) may be approximated by weighted sums of the micrologs. Since the weights depend on the relative importance of the microvariables in the aggregates, in general we shall have $\lambda_i/\mu_i \neq \lambda_j/\mu_j$, for some i and j. Lastly, in that paper it is proved that the approximated aggregate variables are cointegrated by the vector $(1 - c)$.

Now, if the linear approximating model is considered as an autonomous (linear) model, one may argue, on the basis of the above considerations, that aggregate cointegration depends on the vector

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* The reader can easily check that c can not vanish, because $k = -1$ would imply that λ is cointegrating for $y(t)$; nor can it become infinite because $k = -3/7$ would imply that μ is cointegrating for $x(t)$.

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$\log(x(t))$ having cointegration rank $n - 1$, not on the equality of the micro cointegrating vectors.* However, closer analysis of the logarithmic model leads to the following observations:

(i) The linear approximation mentioned above is no longer valid if we drop any one of assumptions (†), (‡) (this may be easily checked by examining the elementary calculations in Lippi (1988), section 3.4).

(ii) It might be worthwhile examining the log model directly. Here I limit myself to some informal considerations. Firstly, under assumptions (†) and (‡), cointegration of $\log(X(t))$ and $\log(Y(t))$ can be shown even without resorting to any linear approximation. Consider the case $n = 2$. Cointegration of the macro logs is equivalent to the existence of a real $\tilde{c} \neq 0$ such that the following ratio:

$$R(t) = \frac{y_1(t) + y_2(t)}{(x_i(t) + x_2(t))^{\tilde{c}}}$$

is stationary. Using (†):

$$R(t) = \frac{x_1(t)^c u_1(t) + x_2(t)^c u_2(t)}{(x_1(t) + x_2(t))^{\tilde{c}}},$$

where $u_i(t)$ is $I(0)$. Putting $\tilde{c} = c$ and using (‡):

$$R(t) = \frac{u_1(t) + u_2(t)v(t)^c}{(1 + v(t))^c},$$

where $v(t)$ is $I(0)$.

(iii) On the other hand, if condition (‡) were valid but we had, instead of (†):

$$\log(y_i(t)) = c_i \log(x_i(t)) + I(0),$$

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* Incidentally, as the above example shows, if $\lambda_i/\mu_i \neq \lambda_j/\mu_j$, for some i and j, equality of the c_i's is not sufficient for the the result $c = c_i$, where c is the macro cointegrating coefficient. The latter equality occurs only with particular cointegrating vectors.

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with $c_1 \neq c_2$, we would find:

$$R(t) = \frac{x_1(t)^{c_1 - \tilde{c}} u_1(t) + x_1(t)^{c_2 - \tilde{c}} u_2(t) v(t)^{c_2}}{(1 + v_t)^{\tilde{c}}},$$

which can not be stationary, for any value of $\tilde{c}$.

(iv) The above analysis, though not very rigorous, shows that in the non-linear logarithmic case cointegration is kept through aggregation under conditions that are stricter than in the linear case.

(v) However, condition (†) with $c = 1$, and condition (‡), taken together, hold in the multiplicative counterpart of linear models in which no condition holds for the micro cointegrating vectors, while $x(t)$ fulfills the condition of case (d):

$$x_i(t) = \alpha_{ij} x_j(t) + I(0),$$

with $\alpha_{ij} \neq 0$, for any i and j. When the $I(0)$ term is written in the multiplicative form the latter equations mean the stationarity of the ratios $x_i(t)/x_j(t)$. On the other hand, when rewritten in multiplicative form, the linear cointegration equations:

$$y_i(t) = p_i x_i(t) + I(0)$$

assume the shape (†), with $c = 1$; in other words, the multiplicative version of these equations retains the unit long-run elasticity of $y_i(t)$ with respect to $x_i(t)$, so that the cointegrating coefficients for the log equations will all be equal to one, independently of the differences among the individual long-run "propensities" p_i.

In conclusion, case (a), i.e. equality of the micro cointegrating vectors, implies condition (1) and therefore macro cointegration when $\lambda = d\mu$ (d scalar). However, if $\lambda_i/\mu_i \neq \lambda_j/\mu_j$, for some i and j, then case (a) implies that condition (1) does not hold; with such aggregating vectors condition (1) would only hold if the c_i's were not equal and a

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particular relationship linking the coefficients c_i, μ_i, λ_i were fulfilled; but since the vectors λ and μ are not intrinsic to the economic reality represented by the c_i's, it seems that such a relationship can not be given any economic meaning.

As the result of a rather unrigorous analysis of logarithmic aggregation, I observe that equality of the micro cointegrating vectors (cointegrating the micro logs) and stationarity of the ratios $x_i(t)/x_j(t)$ are sufficient for cointegration. I conjecture that both such conditions are also necessary.

4. Analysis of case (a) is completed. Let us now turn to cases (b) and (c). Remember that case (b) is a degenerate subcase of (c). Therefore, by Facts 2 and 3, cases (b) and (c) (or (d)) imply macro cointegration.

One reason to take (b) into particular consideration is, as shown in Lippi (1988) (section 5.1), that under (b) the dynamic relationship between X_t and Y_t can be obtained by elementary methods. However, if the cointegration-aggregation issue is considered by itself, there is no reason to pay special attention to case (b). Case (c) is not only sufficient for macro cointegration, it also has a clear economic meaning and, as already observed in section 1, it is empirically more appealing than case (b).

Moreover, I claim that if condition (1) does not hold, and if we exclude cases which do not appear to possess economic meaning, (c) is also necessary for cointegration of the aggregates. In fact, if condition (1) does not hold, and we are not in case (c), a non-empty intersection between S and V_x can occur only if $0 < \dim(V_x) < n-1$ and a non-linear restriction links the coefficients c_i, λ_i, μ_i to the cointegrating vectors of $x(t)$. Since the vectors λ and μ—independently of whether the condition $\lambda = d\mu$ holds or not—are not intrinsic to the economic reality represented by $x(t)$ and $y(t)$, the above restriction will be void of any economic meaning.

The case $n = 3$ will be sufficient to clarify the point. In this case $0 < \dim(V_x) < n-1$ means $\dim(V_x) = 1$. Assume that $K = (k_1 \ k_2 \ k_3)$ is the cointegrating vector for x_t (K is unique up to a factor; at least

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two of the components of K do not vanish). If condition (1) does not hold and the macro variables are cointegrated by $(1 - c)$, the vector

$$(\lambda_1 c_1 - \mu_1 c \quad \lambda_2 c_2 - \mu_2 c \quad \lambda_3 c_3 - \mu_3 c)$$

must be cointegrating for x_t. Therefore:

$$\lambda_1 c_1 - \mu_1 c = h k_1$$
$$\lambda_2 c_2 - \mu_2 c = h k_2$$
$$\lambda_3 c_3 - \mu_3 c = h k_3$$

for some $h \neq 0$, $c \neq 0$. This is possible only if:

$$\det \begin{pmatrix} \mu_1 & k_1 & \lambda_1 c_1 \\ \mu_2 & k_2 & \lambda_2 c_2 \\ \mu_3 & k_3 & \lambda_3 c_3 \end{pmatrix} = 0.$$

Thus we find ourselves in a situation which is typical of aggregation problems. Aggregation of cointegrated vectors retains cointegration if, and only if, either a particular condition on the "microbehaviors" holds, i.e. condition (1), that is case (a) when $\lambda = d\mu$; or the microvariables on the left-hand side of the microequations fulfill a restriction which is independent from the microequations, i.e. case (c). Mixed cases do not seem to deserve interest.

5. As already stated, case (c) (or (d)) has a clear economic meaning. But this does not mean that occurrence of (c) in empirically relevant situations is likely.

Let us consider the consumption-income example. Case (c) may be rephrased either by saying that there exists only one $I(1)$ source of variation for all microincomes; or by saying that there exist many $I(1)$ sources of variation for microincomes: i.e.:

$$x_i(t) = \theta_i^{(1)} X^{(1)}(t) + \theta_i^{(2)} X^{(2)}(t) + \cdots + I(0),$$

but that the ratios $\theta_i^{(h)}/\theta_i^{(k)}$ do not vary across agents. Neither of these assumptions appear to be credible for real economies.

A strong simplification of the issue can result from the distinction between common factors and those variables which affect only individual microincomes. Following Granger (1987), we might conclude that when we take into account the immense size of real economies, only the variables $X^{(h)}(t)$ that are common factors are relevant for the aggregates. But, even so, can we affirm that there exists only one $I(1)$ common factor for all the microincomes? Or that the condition $\theta_i^{(h)}/\theta_i^{(k)}$ invariant over agents holds if we consider only common factors?

An answer to this question would go far beyond this note. I will only observe, firstly, that the issue will take on very different meanings with the different macroequations, and therefore the different empirical frameworks considered. Secondly, that in many important empirical cases—and the consumption-income example is perhaps the most important—an answer is not possible without specifying a complete model for the economy. And this, I think, is a characteristic of many aggregation problems in economics.

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