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**Repeatable and una tantum real options:  
a dynamic programming approach**

di

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# REPEATABLE AND UNA TANTUM REAL OPTIONS: A DYNAMIC PROGRAMMING APPROACH

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**Abstract.** This paper deals with real shared options, which are divided into two classes: *repeatable* options and *una tantum* options. The former include those real options that any competitor in the same industry can exercise, the latter are single, unique options held by several competitors but exercisable only by one of them. This work especially focuses on the second class and tries to find the optimal exercise rule through a dynamic programming approach rather than a financial one, making use of Poisson jump processes. Different situations are treated, in particular interrelations between proprietary options and shared options are found, depending on the degree of exclusiveness of the option. An example of *una tantum* option is given where the optimal decision rule comes out to be, under certain hypotheses, the standard Net Present Value. A case of sequential investment is also studied and, finally, a connection with *repeatable* options is suggested in order to comprehend more general settings.

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# REPEATABLE AND UNA TANTUM REAL OPTIONS: A DYNAMIC PROGRAMMING APPROACH

## Introduction

The aim of the paper is to give a conceptualization of real shared options, which can affect the value  $V$  of an investment or directly influence the value  $F$  of the option. We first give a conceptual framework for the subdivision of shared options into two classes, namely *repeatable* options and *una tantum* options. We then concentrate on the second class and in section 1. we propose an example of vanishing option which is shown to be an extension of proprietary options and whose value coincides with a particular case already studied in literature. In section 2. we make the assumption that a *una tantum* option can be kept alive by paying a *keeping-alive* cost and in one case the optimal decision rule comes out to be the standard NPV. A little different assumption is taken in section 3. where we suppose that exercise of the option can be forced by competitive interaction. Section 4. is devoted to a coincide exposition of a two-stage investment for a proprietary option, then the same procedure is applied in section 5. for a *una tantum* option. Finally we give some insights about interrelations between *repeatable* options and *una tantum* options by presenting two examples of options which are either *repeatable* and *una tantum*.

## Conceptual framework

An industrial or strategic investment opportunity can be seen as a real option if there is some leeway about the timing of the investment. That is, it is possible to postpone action to get more information about the project. So we can see an option to defer an investment as analogous to a call option on a common stock. Option theory can be applied in order to get the correct value of deferring an investment project (Mc Donald and Siegel (1986) and Trigeorgis (1986)), but there is a remarkable difference between real and financial options: real options can be proprietary, which gives an exclusive right of exercise, or shared, for which the right to exercise is collective. Shared options are the result of competitive interaction; in an industrial framework

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firms have to do with uncertainty about competitors' behaviors that influence, partially or totally, their final payoff. Real options are, *a priori*, always shared, but competitive warfare gives rise to "isolating devices" (Buttignon (1990)) which tend to defend the option and make it as exclusive as possible. Among these we can find proprietary rights like patents, proprietary rights of manufacture etc. or company's exclusive knowledge of a market or unique technology that competitors cannot reach. These can completely protect the exclusiveness of an option to invest. Other isolating devices give just a partial and temporary protection: superior knowledges, economies of scale, learning curves, delay in competitors' reaction etc. On the other hand, competitive dynamics tends to smooth away or even delete their effects as soon as they become shared by several companies and as soon as other isolating devices are put into action by competitors. According to Kester (1984) the timing of exercise of an option depends on its degree of exclusiveness and on the degree of competitive rivalry, but it is quite natural that these two variables are interdependent, in particular the latter can affect the value of the former.

Discounted Cash Flow methods are inadequate to give the correct value of an option to invest, since they don't take into account the dynamical aspect of a decision process, but as for contingent claims analysis, it seems to be itself unable to capture the competitive interrelations, which play an important role in a decision process; furthermore, one should take into consideration the opportunity given to the firm of partially affecting the investment payoff. This opportunity arises thanks to competitive skill and advantages held by the investor. He/She plays upon control variables to determine (at least partially) the value of the investment. In capital markets variables are completely stochastic, in a competitive framework they are partially stochastic and partially determined by competitive rivalry. If we regard the chance of investing as an option and the (expected) net present value of the business as the underlying stock, how do competitive interrelations influence these two variables? We think it is necessary to subdivide shared options into two classes: the first one includes situations in which the option is repeatable by other competitors (investment in research, marketing, vertical and horizontal integration, internationalization etc.) that is several companies hold an option on the same security. In capital markets, options are *repeatable* but they are separated and independent one of another so that an option's payoff

isn't affected by the other options; in an industry each action has indirect effects on the other competitors' payoff. If the same option is exercised by several investors then the value of the business is reduced depending on competitors' power and capacity. In this case, the chance to be the only ones to enter a business can guarantee the exclusivity of the option, but also being the first to enter it can accrue first-mover advantages which allows the pioneer to raise sustainable sources of competitive advantages. For example, as regards technological investments (research and development), pioneers get the opportunity to shape the way a product is defined or marketed in a way that favors them. They can acquire reputation as the pioneers or assure themselves favorable access to facilities or inputs, gain unique channel access for a new product, define the standards for technology or for other activities, forcing later movers to adopt them (Porter (1985)). We call "*repeatable*" this kind of shared options.

The second class of shared options, which we call "*una tantum*", describes those cases in which one single option is written on a business. A plurality of conflicting competitors hold the option, which can be exercised *una tantum* by just one of the holders to the detriment of the others, who are then left standing. An example of this kind of options could be the opportunity, given to all the companies in an industry, of buying a firm. A firm for sale can be seen as an option to future growth and expansion in the market. It is necessary then to calculate the trade-off between the positive value of waiting, which brings additional information about the quality of the business, and the negative value given by the opportunity left to the other holders to exercise the option earlier. This second type of options isn't either available in capital markets.

The conceptual difference between *repeatable* options and *una tantum* options is formally expressed by modifying the value  $V$  of the investment in the former, in order to take into account competitive rivalry; by reducing directly the value  $F$  of the option itself in the latter.

In both classes the value  $F$  of the option is modified: in the *repeatable* options the effect is indirect through a change in  $V$ , due to competitive interaction. In the *una tantum* options  $V$  remains fixed, but  $F$  changes depending on the probability of exercise by another holder. To deal with *repeatable* options we can describe the value  $V$  as a combined geometric

Brownian motion and Poisson jump process. This approach has been followed by Trigeorgis (*cit.*) and recently proposed again by Dixit and Pindyck (1994).

We will focus on *una tantum* options and will take a dynamic programming perspective rather than a financial point of view for the following reasons:

- a) it associates in a natural and intuitive way the decision processes with the goals previously set and the final result;
- b) it is more general because it doesn't require the contingent claims analysis's hypotheses and it isn't based on quadratic utility functions as CAPM is;
- c) it is useful for interpretative aims because it allows to see the dynamic programming criterion as an extension of Net Present Value criterion. As a matter of fact, the resolution of an optimal stopping problem derives from a comparison between two net present values: the first concerns the exercise of the option, the other one relates to the value of waiting (Dixit and Pindyck show that use of dynamic programming in valuing real options is, in a sense, a generalization of Net Present Value rule);
- d) it gives the opportunity to show that, in some circumstances, the Net Present Value rule is still useful in valuing real options;
- e) it allows us to translate formally in a natural way the conceptual difference between the two kind of options through a direct effect on  $F(P)$  (*una tantum*) or on  $V(P)$  (*repeatable*);
- f) it allows to overcome the strong assumption that the value  $V$  of an investment can be spanned by existing assets in capital markets (just think of an R & D venture or a strategic investment).

Thus far we have seen that it is important to evaluate the degree of exclusiveness of a shared option in view of the strict connection with the option value. Suppose that an investment opportunity is available and that there is no fixed finite time horizon for the decision problem. Exercise of the project in hand gives the investor the value  $V$ , which we suppose depending on the price  $P$  so that the holder will get, if investment is undertaken, the perpetual revenue stream  $P$  over each interval  $dt$ . We will assume a continuous setting



and suppose that  $P$  follows a geometric Brownian motion:

$$dP = \alpha P dt + \sigma P dz$$

where  $dz$  is a Wiener Process. If future revenues are discounted at the rate  $\rho$  it is easy to see that the expected present value of the business is  $V(P) = P/\delta$  where  $\delta := \rho - \alpha$ . Let  $I$  denote the investment cost and let  $F(P)$  be the value of a *una tantum* american call option written on a certain business.

### 1. Vanishing una tantum options

Let us suppose that a company holds a *una tantum* option on a certain project and that it can indefinitely defer the decision of investment. Competitive interaction is such that our holder must evaluate the arising trade-off: if the company exercises it gives up the chance to have further information about  $P$ , but if it waits then it runs the risk of missing its option rights because of competitors' exercise and consequent vanishing of the option itself. Examples are purchase of a firm or a land or exclusive agreements with customers, distributors or suppliers. We look for the critical value  $P^*$  for which is optimal to invest immediately. Let us begin considering the range  $P < P^*$  for which waiting is optimal. Let  $\lambda dt$  denote the probability that over the next short interval  $dt$  another holder will exercise the option: in this case, it will vanish and our investor won't be able to exercise it anymore. On the contrary, with probability  $1 - \lambda dt$  the option will be alive with value  $F(P)$ .  $\lambda$  is then the *mean arrival rate* of an "event" in a Poisson jump process. We have

$$F(P) = \frac{1}{1 + \rho dt} \lambda dt \cdot 0 + \frac{1}{1 + \rho dt} (1 - \lambda dt) \mathcal{E}(F(P + dP))$$

that is

$$F(P) = \frac{1}{1 + \rho dt} (1 - \lambda dt) \mathcal{E}(F(P + dP)). \quad (1)$$

Using Ito's Lemma we obtain

$$\frac{1}{2} \sigma^2 P^2 F''(P) + \alpha P F'(P) - (\rho + \lambda) F(P) = 0 \quad (2)$$

whose general solution is  $F(P) = A_1 P^{\beta'_1} + A_2 P^{\beta'_2}$ , where  $\beta'_1$  e  $\beta'_2$  are, respectively, the positive and negative root of the equation

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - (\varrho + \lambda) = 0. \quad (3)$$

Over the range  $P > P^*$  our firm exercises the option and therefore  $F(P) = P/\delta - I$  where  $\delta := \varrho - \alpha > 0$ . For economic reasons we have  $F(0) = 0$  so that  $A_2 = 0$ . The boundary conditions involved are known as *value-matching* condition and *smooth-pasting* (or *tangency*) condition, which express the continuity of  $F(P)$  and  $F'(P)$  in  $P = P^*$ . They become, in this case,

$$A_1(P^*)^{\beta'_1} = \frac{P^*}{\delta} - I$$

$$\beta'_1 A_1(P^*)^{\beta'_1-1} = \frac{1}{\delta}.$$

From these we get the value of  $A_1$  and  $P^*$ ; the latter comes out to be

$$P^* = \frac{\beta'_1}{\beta'_1 - 1} \delta I \quad (4)$$

or else

$$V(P^*) = \frac{\beta'_1}{\beta'_1 - 1} I. \quad (5)$$

If the option is proprietary then  $\lambda = 0$  and we have, over the range  $P < P^*$ ,  $F(P) = AP^{\beta_1}$  where  $\beta_1$  is the positive solution of the equation

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - \varrho = 0. \quad (6)$$

The critical price  $P^*$  is

$$P^* = \frac{\beta_1}{\beta_1 - 1} \delta I. \quad (7)$$

Comparing (6) with (3), we see that in (3) the term  $\lambda$  gets added to  $\varrho$ . Therefore  $\beta'_1 > \beta_1$  and  $\beta'_1/(\beta'_1 - 1) < \beta_1/(\beta_1 - 1)$ . This result is also intuitive

because a shared option is such to suggest an earlier exercise due to risk of option's vanishing. So the price threshold  $P^*$  is reduced.

It should be noted that the result obtained in this case coincides with a particular one considered by Trigeorgis (*cit.*) and, subsequently, by Dixit-Pindyck (*cit.*), where the variable  $V$  follows a combined geometric Brownian motion and Poisson jump process and where the option is repeatable by anyone. The option in hand is therefore what we have called a *repeatable* option. They assume

$$dV = \alpha V dt + \sigma V dz - V dq$$

where  $dq = \varphi$  with probability  $\lambda dt$  and  $dq = 0$  with probability  $1 - \lambda dt$ ,  $0 \leq \varphi \leq 1$ . It means that  $V$  will fluctuate as a geometric Brownian motion, but over each time interval  $dt$  there is a small probability  $\lambda dt$  that it will drop to  $(1 - \varphi)$  times its original value and it will then continue fluctuating until another "event" occurs. The value  $F(P)$  of the option must satisfy

$$\varrho dt F(V) = \mathcal{E}(dF).$$

Using the version of Ito's Lemma for combined Brownian and Poisson processes, they obtain the differential equation

$$\frac{1}{2}\sigma^2 V^2 F''(V) + \alpha V F'(V) - (\varrho + \lambda)F(V) + \lambda F[(1 - \varphi)V] = 0. \quad (8)$$

The solution to (8) is of the form  $F(V) = AV^{\beta_1}$  but now  $\beta_1$  is the positive solution of the nonlinear equation

$$\frac{1}{2}\sigma^2 \beta(\beta - 1) + \alpha\beta - (\varrho + \lambda) + \lambda(1 - \varphi)^\beta = 0. \quad (9)$$

The value of  $\beta_1$  that satisfies both (9) and the condition  $F(0) = 0$  must be found numerically.

In the particular case of  $\varphi = 1$  equations (8) and (9) coincide, respectively, with (2) and (3) and the threshold is just given by (4). Thus two different shared options, a *una tantum* one and a *repeatable* one, can lead to the same result. In the *una tantum* option case we assume that the option itself can

vanish while  $V$  follows a geometric Brownian motion; the *repeatable* case assumes that the option is infinite-lived but the repeatability left to several competitors can drop to zero the value  $V$  of the project. It is extremely intuitive to capture the analogy between the two options: saying that an option vanishes and therefore is impossible to exercise it is like saying that if we exercise the option, then my payoff is zero since the price  $V$  is set to zero.

Finally, it is also patent that this case of option includes the proprietary option case; in fact, letting  $\lambda = 0$  equations (3) and (6) coincide. The introduction of the parameter  $\lambda$  allows us to represent in a simple way the degree of exclusiveness of an option. We actually interpret this parameter both in terms of *mean arrival rate* (in a probabilistic sense) and degree of exclusiveness (in a conceptual sense). Looking back to equation (1) we can also think this way: our decision-maker holds a proprietary option which, with probability  $\lambda dt$ , is snatched away from him/her by a competitor, who will hold it, once stolen, as a proprietary option. This remark gives us the chance to introduce a typology of options which are halfway between proprietary and *una tantum* options.

## 2. *Una tantum* options and keeping-alive cost

Let assume that a *una tantum* option to invest keeps alive with probability  $1 - \lambda dt$  over the next short interval of time  $dt$  but competitors can cause the option to vanish by exercising it with complementary probability. Suppose, for the latter case, that the firm can resist and keep alive the option by defraying the cost  $\bar{I}$ ; it represents an isolating device able to perpetuate the option. Several factors affect  $P^*$  in one way or another:  $\lambda$  denotes the risk of competitors' actions, therefore leads to earlier exercise as well as the foregoing revenue stream  $Pdt$ ; on the contrary, the risk of undertaking an investment, formally expressed by the term  $\sigma Pdz$ , suggests the holder to wait and get more information about  $P$ . The cost  $\bar{I}$  allows to keep alive the option in order to receive information but involves an expense so that it urges to earlier exercise [in general, isolating devices lower the value of  $\lambda$ . It could be interesting to study  $\bar{I}$  as a function of  $\lambda$ : the more intense the competition the greater the cost of putting into action isolating devices such as to avoid option's vanishing].

Suppose, *in primis*,  $\bar{I} < (\varrho/\lambda)I$ . Then, for  $P < P^*$  we have

$$F(P) = \frac{1}{1 + \varrho dt} \lambda dt \mathcal{E}(F(P + dP) - \bar{I}) + \frac{1}{1 + \varrho dt} (1 - \lambda dt) \mathcal{E}(F(P + dP)) \quad (10)$$

and, for  $P \geq P^*$ , we have  $F(P) = V(P) - I = P/\delta - I$ . From (10) we get

$$\varrho dt F(P) = -\lambda dt \bar{I} + \mathcal{E}(dF)$$

and hence

$$\frac{1}{2} \sigma^2 P^2 F''(P) + \alpha P F'(P) - \varrho F(P) - \lambda \bar{I} = 0 \quad (11)$$

The solution to this equation is of the form

$$F(P) = c_1 P^{\beta_1} + c_2 P^{\beta_2} - \frac{\lambda \bar{I}}{\varrho}$$

in which we must have  $c_2 = 0$  in order to avoid the *absurdum*  $F(0) = \infty$  (due to  $\beta_2 < 0$ ). Note also that  $\bar{I}$  causes  $F(P)$  to fall and take negative values over the range  $P < (\lambda \bar{I} / \varrho c_1)^{1/\beta_1}$ . We cannot accept a negative value for an option, so  $F(P)$  becomes:

$$F(P) = \max \left[ 0; c_1 P^{\beta_1} - \frac{\lambda \bar{I}}{\varrho}; \frac{P}{\delta} - I \right].$$

Over the range  $0 \leq P < (\lambda \bar{I} / \varrho c_1)^{1/\beta_1}$  the first term applies; for  $(\lambda \bar{I} / \varrho c_1)^{1/\beta_1} \leq P < P^*$  we have the second one; for  $P \geq P^*$  we use the third one (see fig. 1).

The boundary conditions (*value-matching* condition and *smooth-pasting* condition) lead to

$$P^* = \frac{\beta_1}{\beta_1 - 1} \delta \left( I - \frac{\lambda \bar{I}}{\varrho} \right). \quad (12)$$

Compared with proprietary options the multiplicative factor remains unvaried but the base  $I$  is now changed to  $(I - \frac{\lambda \bar{I}}{\varrho})$ . As a matter of fact, proprietary options represent a particular case of these letting  $\lambda = 0$  or

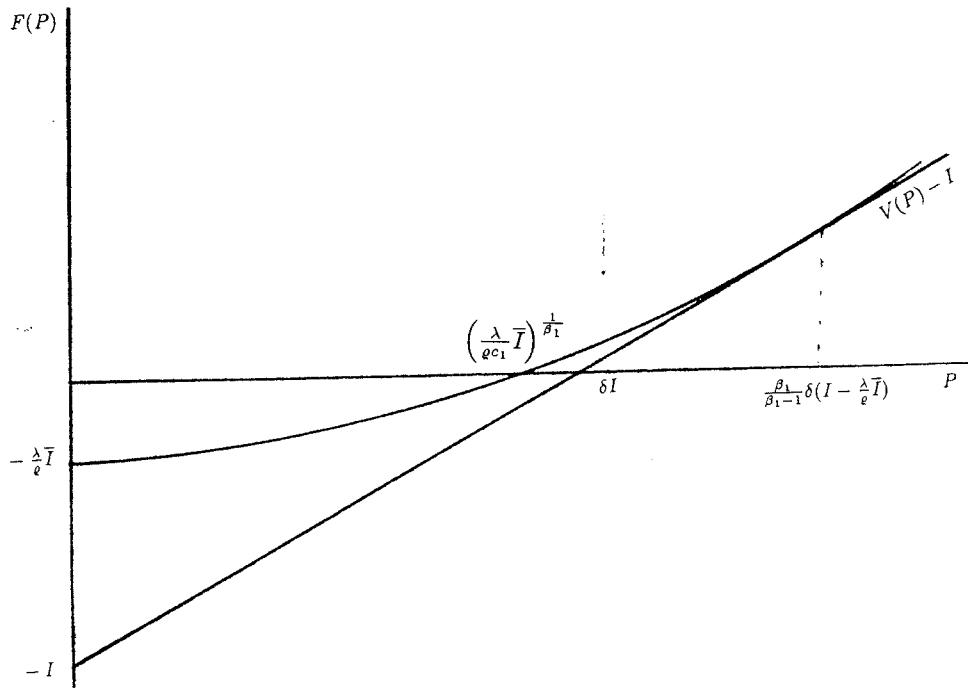


FIGURE 1. The case of  $\bar{I} < \frac{\varrho}{\lambda} I$

$\bar{I} = 0$ . It is clear that the greater is the *keeping-alive* cost the smaller is the threshold  $P^*$ ; from (12) we get

$$\frac{dP^*}{d\bar{I}} = -\frac{\beta_1}{\beta_1 - 1} \frac{\delta \lambda}{\varrho} < 0.$$

When  $\bar{I} = 0$  we have a proprietary option and therefore it is natural that waiting is at its highest value; as soon as  $\bar{I}$  increases the price threshold  $P^*$  decreases (see fig. 2).

This was over the range  $\bar{I} < (\varrho/\lambda)I$ . But what if  $\bar{I} \geq (\varrho/\lambda)I$ ? We have two cases: consider first  $\delta I < (\lambda \bar{I}/\varrho c_1)^{1/\beta_1}$  (see fig. 3a). In this case over the range  $0 \leq P < \delta I$  the two functions  $P/\delta - I$  and  $c_1 P^{\beta_1} - (\lambda/\varrho)\bar{I}$  are both negative. Therefore it is natural to let  $F(P) = 0$ . For  $\delta I \leq P < P^*$  we choose, in view of the graphs,  $F(P) = V(P) - I$ . Let now be  $P \geq P^*$ : if we choose  $F(P) = c_1 P^{\beta_1} - (\lambda/\varrho)\bar{I}$  we get to the absurd that it is never worthwhile to exercise the option and, moreover, the greater is  $P$  the greater is

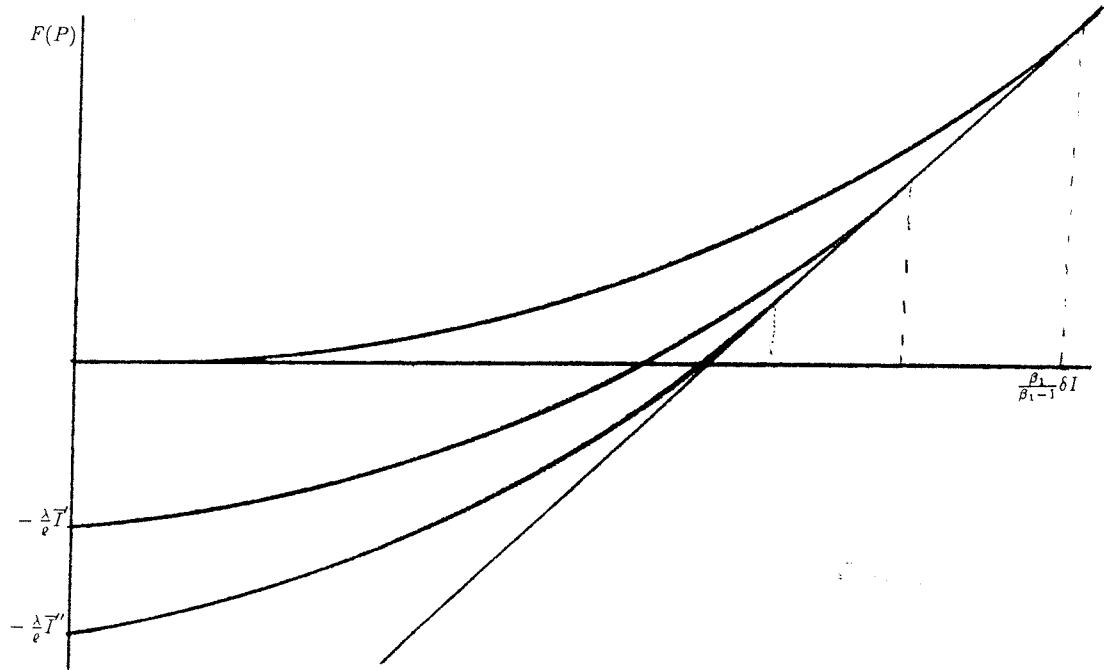


FIGURE 2. Relation between  $\bar{I}$  and  $P^*$

the value of waiting. But this doesn't make any sense economically: at some high enough price, the opportunity cost of foregone profit flow becomes great enough to suggest exercise of the option. Then we choose again  $F(P) = V(P) - I$  for  $P > P^*$ .

As for the case in fig. 3b it is obvious that  $F(P) = 0$  when  $0 \leq P < (\lambda \bar{I} / \rho c_1)^{1/\beta_1}$ , whereas over the range  $(\lambda \bar{I} / \rho c_1)^{1/\beta_1} \leq P < \delta I$  the graph of the exponential function lies above  $V(P) - I$ . This should suggest to wait and not to exercise, but the same decision rule is adopted if we apply standard NPV criterion. As regards the interval  $[\delta I, +\infty)$  we have to look at the function  $V(P) - I$ , according to what we explained earlier.

Finally, the decision criterion to be applied over the range  $\bar{I} \geq (\rho/\lambda)I$  is our dear old NPV again. The significance of such a conclusion is interesting: if the cost of isolating devices is too high, the option isn't an option anymore and what we have to do is investing at once if the Net Present Value is positive. The term "too high" has a quantitative translation into the factor  $(\rho/\lambda)I$ . Therefore  $\rho/\lambda$  is the maximum sustainable cost factor beyond which

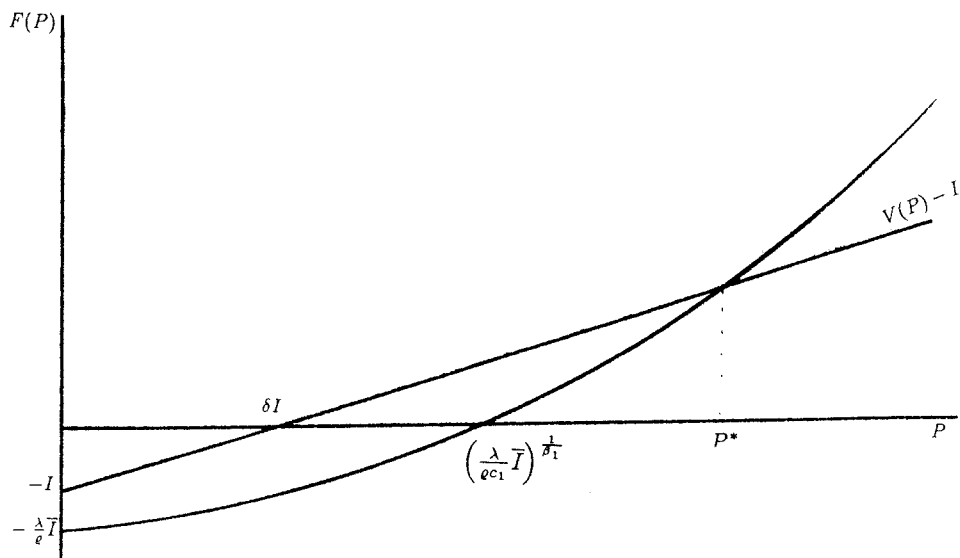


FIGURE 3a. The case of  $\delta I < (\frac{\lambda}{e c_1} \bar{I})^{\frac{1}{\beta_1}}$

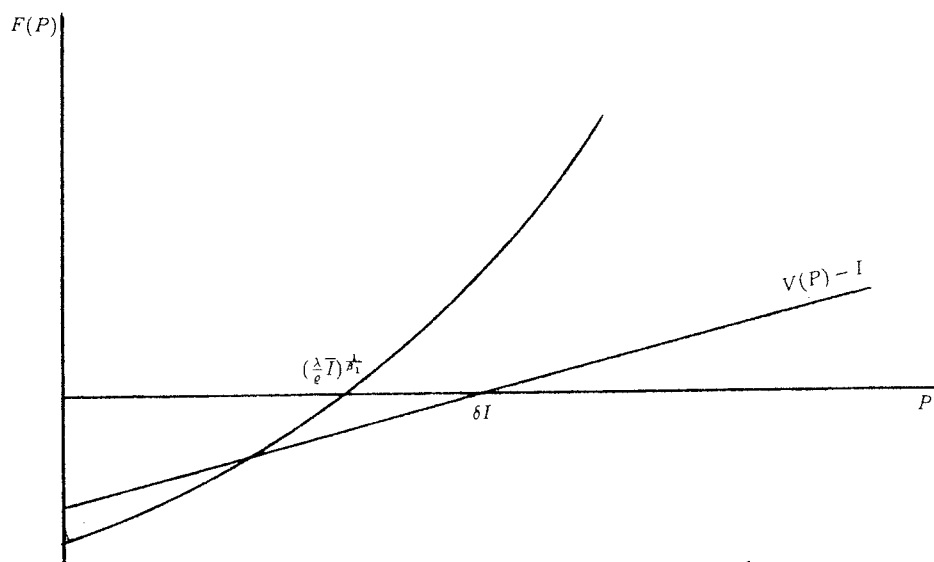


FIGURE 3b. The case of  $\delta I \geq (\frac{\lambda}{e c_1} \bar{I})^{\frac{1}{\beta_1}}$



the option doesn't retain any more value. Below it we use dynamic programming to choose the threshold  $P^*$ , above it we apply standard NPV due to excessive *keeping-alive* cost.

This last result corroborates our decision of taking a dynamic programming perspective rather than an option pricing approach: dynamic programming is in fact an extension of Discounted Cash Flow methods, allows us to turn back again *naturaliter* to NPV rule and is therefore conceptually more significant.

### 3. Una tantum options and random exercise enforcement

In some circumstances it can be impossible to keep alive an option and the holder can be forced to exercise the option so as not to miss it to competitors' advantage. We can think of the launching of a new product: if our firm's competitors are far away to launch likewise their new product or model, then it is possible to wait and get more information. But if competitors are to introduce it in a short time then the company has to exercise the option right away in order to stem competitors' actions. Since the firm has incomplete information about its competitors, it assigns a value to the probability  $\lambda dt$  that, over the next interval  $dt$ , it will have to invest in the project. We have, in the *continuation region*  $P < P^*$ ,

$$F(P) = \frac{1}{1 + \rho dt} \lambda dt \left( \frac{P}{\delta} - I \right) + \frac{1}{1 + \rho dt} (1 - \lambda dt) \mathcal{E}(F(P + dP))$$

and hence

$$\frac{1}{2} \sigma^2 P^2 F''(P) + \alpha P F'(P) - (\rho + \lambda) F(P) + \frac{\lambda}{\delta} P - \lambda I = 0.$$

The solution to this equation is given by

$$F(P) = AP^{\beta_1'} + \frac{\lambda P}{\delta(\delta + \lambda)} - \frac{\lambda I}{\rho + \lambda}.$$

Since we must have  $F(P) \geq 0 \forall P$ , the value of the option is

$$F(P) = \max \left[ 0, AP^{\beta_1'} + \frac{\lambda P}{\delta(\delta + \lambda)} - \frac{\lambda I}{\rho + \lambda} \right],$$

On the other hand we have  $F(P) = V(P) - I$  in the *stopping region*  $P \geq P^*$ . From the usual boundary conditions

$$F(P^*) = V(P^*) - I \quad \text{and} \quad F'(P^*) = V'(P^*)$$

we finally get

$$P^* = \frac{\varrho}{\varrho + \lambda} \frac{\beta_1'}{\beta_1' - 1} \delta' I, \quad \delta' := (\delta + \lambda) \quad (13)$$

(see also fig. 4).

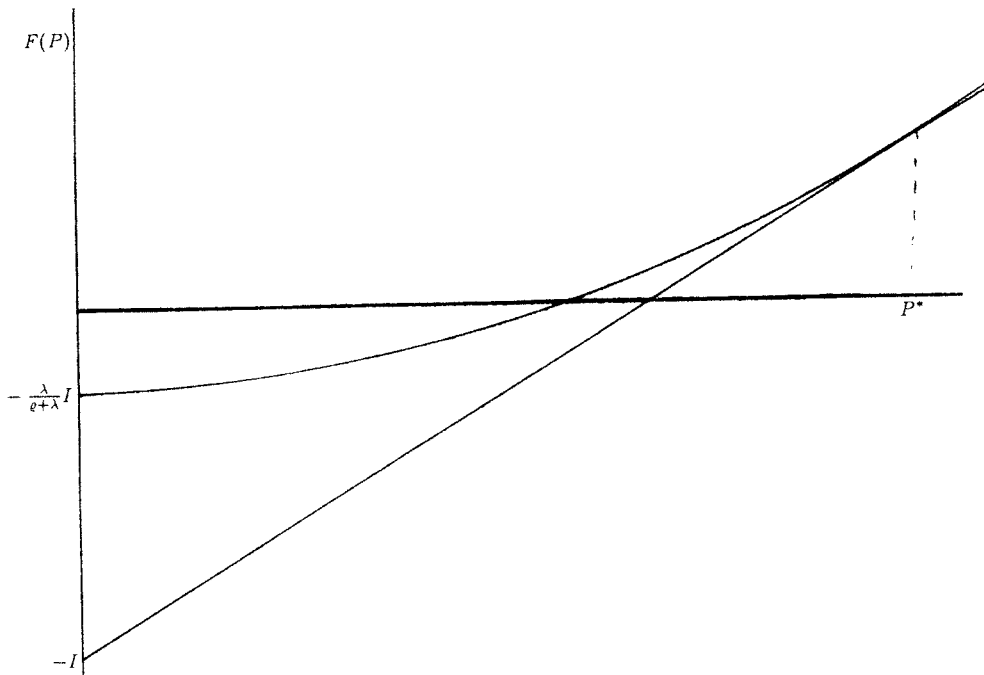


FIGURE 4. Option value with random exercise enforcement

Compared with the case studied in 1. the term  $\delta$  is changed to  $\delta'$  and the new multiplier  $\varrho/(\varrho + \lambda)$  comes out. The latter tends to lower  $P^*$ , the former tends to raise it. Which one of the two prevails? In the case studied in 1. we had a probability of option's vanishing. Here our holder can only be enforced to a premature exercise and can therefore wait until a higher price is reached.

We would then expect, for this case, an increase of  $P^*$ . We can verify the intuition by studying the ratio between (13) and (4). It is

$$f(\lambda) = \frac{\varrho}{(\varrho + \lambda)} \frac{(\varrho + \lambda - \alpha)}{\varrho - \alpha};$$

since

$$f'(\lambda) > 0 \quad \forall \lambda > 0,$$

and  $f(0) = 1$ , then the threshold the function is increasing and the critical value  $P^*$  in (13) is higher than in (4).

#### 4. Proprietary options and sequential investment

Let us now turn for a while to proprietary options in order to examine sequential investment and, in particular, a two-stage investment where we suppose that the firm doesn't earn any cash flow from the project until the project is completed. We will refer to Dixit and Pindyck (*cit.*) for all results of this section. Our firm can start undertaking the project while observing the evolution of price dynamics. If price moves down the company can temporary stop the project and wait for a higher level of  $P$ . In the following example, implementation is split into two stages, so that the first stage is undertaken if  $P$  reaches the critical value  $P_1^*$ ; then the second stage is completed if  $P$  reaches the threshold  $P_2^*$  and the project starts then yielding the profit flow  $P$  over each short interval  $dt$ . An  $n$ -stage investment,  $n = 3, 4, \dots$ , is a simple extension of this case. We work backward by calculating  $F_2(P)$  and the relative value  $P_2^*$ . The value of the option must satisfy the differential equation

$$\frac{1}{2}\sigma^2 P^2 F_2''(P) + \alpha P F_2'(P) - \varrho F_2(P) = 0 \quad (14)$$

subject to the boundary conditions

$$F_2(0) = 0$$

$$F_2(P_2^*) = V(P_2^*) - I_2$$

$$F_2'(P_2^*) = V'(P_2^*)$$

where  $I_2$  is the last tranche of the total investment cost  $I = I_1 + I_2$ . We then obtain  $F(P) = D_2 P^{\beta_1}$  for  $P < P_2^*$ , and  $F(P) = V(P) - I_2$  for  $P \geq P_2^*$ .  $P_2^*$  is given by

$$P_2^* = \frac{\beta_1}{\beta_1 - 1} \delta I_2.$$

We can now back up to the first stage and find  $F_1(P)$  which will also satisfy (14) but now subject to

$$F_1(0) = 0$$

$$F_1(P_1^*) = F_2(P_1^*) - I_1$$

$$F_1'(P_1^*) = F_2'(P_1^*).$$

It is possible to demonstrate that  $P_1^* > P_2^*$  so that in the second one of the three conditions above we have  $F_2(P_1^*) = D_2(P_1^*)^{\beta_1}$ . The solution takes the usual form  $F_1(P) = D_1 P^{\beta_1}$ . Then we easily have

$$P_1^* = \frac{\beta_1}{\beta_1 - 1} \delta (I_1 + I_2).$$

Since  $P_1^* > P_2^*$  we would expect the investment to be implemented at once, combining the two stages together by sustaining the total expense of  $I = I_1 + I_2$ . But, first of all, investing takes time and it is often impossible to complete jointly both stages. Also, during implementation of the first stage price can drop below  $P_2^*$ ; consequently, the firm must wait until the threshold  $P_2^*$  is reached again. Our company could even decide to sell the partially completed project to another. Furthermore, there can be legal regulations which impose delay of later stages. We stress that the condition  $P_1^* > P_2^*$  can also be viewed as the result of psychological reasons: once completed the first stage, our firm has invested the quantity  $I_1$ ; if we suppose that the first-stage investment is irreversible, then the negative cash flow  $I_1$  cannot be recovered by selling the project. So the company is more inclined to complete it even if the project value should drop to a lower unexpected value. In fact, if the company didn't go on with the later stage, the effort of entering the first stage would be unuseful. The psychological reason is just this: if one is halfway there, then it doesn't pay to leave work half done.

We will adopt this very way of reasoning in the next section studying the case of *una tantum* options and assume it as a starting condition for the resolution of the problem. It means then that the holder of the option will accept a lower value  $P_2^* < P_1^*$  for the optimal exercise rule.

### 5. Una tantum options and sequential investment

Consider a two-stage investment where the second stage is analogous to the one explained in 3., so that the critical value  $P_2^*$  is also given by (13). We rewrite it here with the appropriate indexes for convenience of the reader:

$$P_2^* = \frac{\varrho}{\varrho + \lambda} \frac{\beta_1'}{\beta_1' - 1} \delta' I_2, \quad \delta' := (\delta + \lambda). \quad (13 - \text{bis})$$

Let us back up to the first stage. Suppose that the firm can be forced, with probability  $\lambda dt$ , to enter the second stage during the next interval  $dt$  due to competitive interaction. As explained earlier we assume  $P_1^* > P_2^*$ . If  $P < P_1^*$  the firm won't exercise unless it is forced to do it, which has probability  $\lambda dt$ . In this case it will invest an amount of  $I_1$  and will come into possession of the option to invest in the second stage. Then

$$F_1(P) = \frac{1}{1 + \varrho dt} \lambda dt \mathcal{E}(F_2(P + dP) - I_1) + (1 - \lambda dt) \frac{1}{1 + \varrho dt} \mathcal{E}(F_1(P + dP)). \quad (15)$$

On the other hand, for  $P \geq P_1^*$ , we have  $F_1(P) = F_2(P) - I_1$ . As we can see,  $F_1(P)$  incorporates  $F_2(P)$  in both cases; it is then necessary to know which one of the two possible expressions for  $F_2(P)$  applies, in accordance with our hypothesis  $P_1^* > P_2^*$ . Let us begin considering the range  $P \leq P_2^* < P_1^*$ . In this region  $F_1(P)$  must satisfy the differential equation

$$\frac{1}{2} \sigma^2 P^2 F_1''(P) + \alpha P F_1'(P) - (\varrho + \lambda) F_1(P) + \lambda (F_2(P) - I_1) = 0 \quad (16)$$

where

$$F_2(P) = A_2 P^{\beta_1'} + \frac{\lambda P}{\delta(\delta + \lambda)} - \frac{\lambda I_2}{\varrho + \lambda}.$$

Let now  $P_2^* < P < P_1^*$ . We have  $F_2(P) = P/\delta - I_2$  and therefore equation (15) becomes

$$F_1(P) = \lambda dt \frac{1}{1 + \varrho dt} \left( \frac{P}{\delta} - I_2 - I_1 \right) + (1 - \lambda dt) \frac{1}{1 + \varrho dt} \mathcal{E}(F_1(P + dP)) \quad (17)$$

Finally, for  $P \geq P_1^* > P_2^*$  we choose

$$F_1(P) = F_2(P) - I_1 \quad \text{and} \quad F_2(P) = \frac{P}{\delta} - I_2. \quad (18)$$

To the end of finding out the value of  $P_1^*$  only the second and third region are relevant, so we have no concern with (16) and concentrate on (17) and (18). From (17) we obtain

$$F_1(P) = A_1 P^{\beta'_1} + \frac{\lambda P}{\delta(\delta + \lambda)} - \frac{\lambda(I_2 + I_1)}{\varrho + \lambda}$$

and from (18) we find

$$F_1(P) = \frac{P}{\delta} - (I_1 + I_2).$$

The continuity of  $F_1(P)$  and the tangency condition in  $P = P_1^*$  give us the value of  $A_1$ :

$$A_1 = \frac{1}{\beta'_1(P_1^*)^{\beta'_1-1}(\delta + \lambda)}$$

from which we get

$$P_1^* = \frac{\varrho}{\varrho + \lambda} \frac{\beta'_1}{\beta'_1 - 1} (\delta + \lambda) I \quad I := I_2 + I_1. \quad (19)$$

This expression is almost identical to (13)-bis, but now the term  $I_1$  gets added to the cost  $I_2$  raising the value of the price threshold.

In a particular case we can demonstrate that  $P_1^* > P_2^*$  without assuming it as a hypothesis of individual behavior. Suppose that our two-stage investment option is such that, if  $P < P_1^*$ , the firm keeps on holding the option to invest in the first stage with probability  $1 - \lambda dt$ , whereas there is a small probability  $\lambda dt$  for the firm to be forced to invest in the entire project at once, (that is, to combine the two stages) with probability  $\lambda dt$ , even if  $P < P_2^*$ . We also assume, as natural,  $\varrho > 0$ . Thus, over the range  $P < P_1^*$  we have

$$F_1(P) = \frac{1}{1 + \varrho dt} \lambda dt \left( \frac{P}{\delta} - I \right) + (1 - \lambda dt) \frac{1}{1 + \varrho dt} \mathcal{E}(F_1(P + dP)).$$

We now use a *reductio ad absurdum* letting  $P_1^* \leq P_2^*$ . Then

$$F_1(P) = A_1 P^{\beta_1} + \frac{\lambda P}{\delta(\delta + \lambda)} - \frac{\lambda I}{\varrho + \lambda}$$

for  $P < P_1^* \leq P_2^*$  and

$$F_1(P) = F_2(P) - I_1 = A_2 P^{\beta_1} + \frac{\lambda P}{\delta(\delta + \lambda)} - \frac{\lambda I_2}{\varrho + \lambda} - I_1$$

for  $P_1^* \leq P \leq P_2^*$ . In accordance with the *tangency* condition we must have

$$\frac{d}{dP} F_1(P_1^*) = \frac{d}{dP} (F_2(P_1^*) - I_1)$$

from which we get  $A_1 = A_2$ . This implies, along with  $F_1(P_1^*) = F_2(P_1^*) - I_1$ ,

$$\frac{\lambda}{\varrho + \lambda} I = \frac{\lambda}{\varrho + \lambda} I_2 + I_1$$

that is

$$\frac{\lambda}{\varrho + \lambda} (I - I_2) = I_1,$$

which is an *absurdum*.

If we then suppose that the firm can speed up the carrying of the project only by investing an additional amount of  $I^\circ - I$ , where  $I^\circ$  represents the total investment for the entire project, then it is possible to demonstrate the thesis  $P_1^* > P_2^*$  with the hypothesis

$$\varrho \neq \frac{\lambda(I^\circ - I_2)}{I_1} - \lambda.$$

We only need to run through the same steps replacing  $I$  by  $I^\circ$ .

## 6. Repeatable and una tantum options: some insights

*Repeatable* options are such that competitive interaction affect the value  $V(P)$  so that it is quite unrealistic to assume a geometric Brownian motion. It should be more satisfying to find a way to describe, at least approximately, the economic conflict among competitors. Trigeorgis and Dixit-Pindyck (*cit.*) suggest to make use of both geometric Brownian motion and Poisson jump processes. Let us consider a particular case of an option which is at the same time *repeatable* and *una tantum*. The option's holder runs the risk of having the option snatched away by competitors (see option vanishing *sub* 1.). This event occurs with probability  $\lambda_2 dt$  over the next short interval  $dt$ . Also, if the firm exercises its option, competitive rivalry tends to drop the value of the project  $V$  to  $(1 - \varphi)$  times its original value with a small probability  $\lambda_1 dt$ , otherwise it will continue fluctuating as a geometric Brownian motion. Letting  $V$  be directly our variable we have to find the value of the option

$$F(V) = \max \left[ V - I, \frac{1}{1 + \rho dt} (1 - \lambda_2 dt) \mathcal{E} (F(V + dV)) \right]; \quad (20)$$

subject to

$$dV = \alpha V dt + \sigma V dz - V dq$$

where  $dq = \varphi$  with probability  $\lambda_1 dt$  and  $dq = 0$  with probability  $1 - \lambda_1 dt$ ,  $0 \leq \varphi \leq 1$ . Applying Ito's Lemma and rearranging terms we obtain the differential equation

$$\frac{1}{2} \sigma^2 V^2 F''(V) + \alpha V F'(V) - \lambda_1 (F(V) - F[(1 - \varphi)V]) - (\rho + \lambda_2) F(V) = 0. \quad (21)$$

The solution to (21) is again of the form  $F(V) = AV^{\beta_1''}$  and  $\beta_1''$  is now the positive solution of the nonlinear equation

$$\frac{1}{2} \sigma^2 \beta(\beta - 1) + \alpha \beta - (\rho + \lambda_1 + \lambda_2) + \lambda_1 (1 - \varphi)^\beta = 0. \quad (22)$$

The equation that satisfies both (22) and the condition  $F(0) = 0$  must be found numerically. If we let  $\varphi = 1$  then equation (22) becomes

$$\frac{1}{2} \sigma^2 \beta(\beta - 1) + \alpha \beta - (\rho + \lambda_1 + \lambda_2) = 0. \quad (23)$$



This equation looks like (3) except that the mean arrival rates  $\lambda_1$  and  $\lambda_2$  are involved. Therefore the critical value  $V^*$  is given by

$$V^* = \frac{\beta_1''}{\beta_1'' - 1} I \quad (24)$$

Let us now consider the case of an option with *keeping-alive* cost and  $\bar{I} < (\varrho/\lambda_2)I$ ; it is easy to derive, from (10), the differential equation

$$\varrho F(V) = \frac{1}{2} \sigma^2 V^2 F''(V) + \alpha V F'(V) - \lambda_1 (F(V) - F[(1 - \varphi)V]) - \lambda_2 \bar{I}. \quad (25)$$

The solution is of the form

$$F(V) = AV^{\beta_1'} - \frac{\lambda_2 \bar{I}}{\varrho}$$

where  $\beta_1'$  is now the positive solution to

$$\frac{1}{2} \sigma^2 \beta(\beta - 1) + \alpha \beta - (\varrho + \lambda_1) + \lambda_1 (1 - \varphi)^\beta = 0. \quad (26)$$

The critical value is, in the case of  $\varphi = 1$ ,

$$V^* = \frac{\beta_1'}{\beta_1' - 1} \left( I - \frac{\lambda_2 \bar{I}}{\varrho} \right) \quad (27)$$

which is, as we can see, analogous to (12).

### Conclusions

In this paper we showed how real shared options can be treated applying dynamic programming and Poisson jump processes, in particular we proposed some examples of *una tantum* options noting that the dynamic programming approach can in some cases turn back to the standard NPV rule. Finally we gave some insights for interrelations between the two classes of shared options making use of two jump processes. We also remarked the strong analogies among the results obtained for the different cases, each one of which can be seen in different manners and as a particular or general case of another.

Future researches could even lead to overcome the concept of shared options. Either an option is shared or not, competition does its work through threats and retaliations, barriers and obstacles, conflicts and cooperation. In any case the firm holding the option is in a dynamic *competitive* environment where several policy makers face one another in order to obtain the best results to the detriment of the others. It is important then to take into account decision-maker's possible action and competitors' reaction. Magni (1995) suggests to consider the value  $V(P)$  as the result of a stochastic optimal control or, even better, the result of a non-cooperative dynamic game. From this point of view the stochastic optimal problem (or the dynamic non-cooperative game) is incorporated into an optimal stopping problem.

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