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Reduction of the Three-Partition Problem

by

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Abstract

The three-partition problem is one of the most famous strongly NP-complete combinatorial problems. We introduce properties which, in many cases, can allow either a quick solution of an instance or a reduction of its size. The average effectiveness of the properties proposed is tested through computational experiments.

Keywords: partition, combinatorial properties, reduction.

1 Introduction

In the *Three-Partition Problem* (3-PARTITION) we are given a positive integer b and a set $N = \{1, 2, \dots, n\}$ of $n = 3m$ elements, each having a positive integer *size* a_j, such that $\sum_{j=1}^{n} a_j = mb$. The problem is to determine whether a partition of N into m subsets, each containing exactly three elements from N and such that the sum of the sizes in each subset is b, exists. The solution is *yes* if such a partition exists, and *no* otherwise.

This is one of the most famous NP-complete problems and, to our knowledge, the first one which was shown to be strongly NP-complete with a non trivial proof (see Garey and Johnson [2], [4]). Indeed an instance I of 3-PARTITION can be solved in polynomial time if the magnitude of the largest integer value, $\mu(I)$, is bounded by a constant, but it remains NP-complete if $\mu(I)$ is bounded by a nonconstant polynomial in the instance size. 3-PARTITION has been frequently used in the literature to prove strong NP-completeness results, especially, right from the beginning, in the area of scheduling theory (see, e.g., Garey, Johnson and Sethi [3] or Lenstra, Rinnooy Kan and Brucker [7]).

In this paper we study combinatorial properties of 3-PARTITION and show that effective techniques can be implemented to substantially reduce the size of an instance and, in many cases, even to solve it. To our knowledge, this is the first analysis of the average difficulty of the problem. In Section 2 we examine possible transformations of the problem into other combinatorial optimization problems. In Section 3 we present properties which can allow one either to establish that an instance has solution *no* or to permanently assign groups of three elements to subsets. An efficient implementation of such properties is described in Section 4, and computationally tested in Section 5.

2 Problem Transformations

An alternative formulation of 3-PARTITION is frequently encountered in the literature: it is not explicitly required that each subset contain three elements, but the sizes are restricted to values satisfying

$$\frac{b}{4} < a_j < \frac{b}{2} \quad (j = 1, \dots, n) \tag{1}$$

(see Garey and Johnson [5]). It is easily seen that (1) implies that in any feasible solution there are exactly three elements in each subset.

An instance with unrestricted sizes can be transformed into an equivalent instance satisfying (1) by defining

$$\overline{a}_j = a_j + b \;\; (j = 1, \dots, n) \;\text{ and }\; \overline{b} = 4b \tag{2}$$

It is immediately seen that there is a one-to-one correspondence between feasible partitions of the two instances. This provides a way of transforming a general instance of 3-PARTITION into an equivalent instance of classical combinatorial optimization problems, as shown below.

Given n elements, each having a positive size, and an unlimited number of bins of identical capacity, the *Bin Packing Problem* (BPP) is the optimization problem consisting in the assignment of all the elements to the minimum number of bins in such a way that the total size in each bin does not exceed its capacity. Given any instance of 3-PARTITION, defined by (a_j) and b, we can define an instance of BPP having the sizes $\overline{a}_j$ and the bin capacity $\overline{b}$ given by (2): if the optimal solution to BPP requires exactly m bins, then we know that the answer to the 3-PARTITION instance is *yes*, and otherwise it is *no*.

(Observe that this would not be true for a BPP instance defined by unrestricted sizes a_j and bin capacity b, since a solution requiring m bins could pack more or less than three elements in some bin.) Since effective exact and approximation algorithms and codes are available for BPP (see, e.g., Martello and Toth [8]), this transformation gives a possible tool for the solution of 3-PARTITION.

An alternative transformation ("dual" to the previous one) can be obtained through multiprocessor scheduling. Given n jobs, each having a positive processing time, and m parallel identical processors which can perform one job at a time, the *Multiprocessor Scheduling Problem* (usually denoted by $P||C_{\max}$ in the scheduling literature, see, e.g., Hoogeveen, Lenstra and van de Velde [6]) requires an assignment of the jobs to the processors, which minimizes the largest completion time of a job (*makespan*). An instance of 3-PARTITION has answer *yes* if and only if the optimal solution to the $P||C_{\max}$ instance defined by processing times $\overline{a}_j$ $(j = 1, \dots, n)$ has makespan equal to $\overline{b}$ (see (2)). An effective exact algorithm for $P||C_{\max}$ was presented by Dell'Amico and Martello [1].

A third possible transformation involves the *0-1 Multiple Knapsack Problem* (MKP), for which algorithms and codes are given in Martello and Toth [8]: given n items, each having an associated profit and weight, and m knapsacks having capacities c_i $(i = 1, \dots, m)$, it is required to select m subsets of items (one per knapsack) such that the total weight in each knapsack does not exceed the corresponding capacity and the total profit of the selected items is a maximum. An instance of 3-PARTITION can be transformed into an equivalent instance of MKP by setting the item weights to $\overline{a}_j$ $(j = 1, \dots, n)$, the item profits to any positive value and $c_i = \overline{b}$ $(i = 1, \dots, m)$: the answer is then *yes* if and only if the optimal solution to the MKP instance has value equal to the sum of all profits.

As will be seen in Section 5, the transformations above are however quite ineffective, on average, for the solution of 3-PARTITION. In the next sections we introduce reduction properties which computationally prove to be extremely useful for the problem.

3 Reduction Properties

In this section we give properties which, in certain situations, either prove that the instance has solution *no* or allow the instance to be reduced. We say that an instance can be *reduced* if we can decide that three specific elements can be assigned to a subset, ensuring that the residual problem has the same solution as the original one.

We assume in the following that the elements are sorted so that

$$a_1 \geq a_2 \geq \ldots \geq a_n \tag{3}$$

We will say that an element k and a pair of distinct elements (i, j) with $i, j \neq k$ are *matchable* if $a_i + a_j + a_k = b$. Similarly, three sizes (not necessarily different from each other) are matchable if there are three matchable elements having such sizes.

3.1 Simple properties

Property 1 *If $a_1 + a_{n-1} + a_n > b$ or $a_1 + a_2 + a_n < b$, then the instance has solution* no.

Proof. In the first case no pair is matchable with element 1, since $a_i + a_j \geq a_{n-1} + a_n$ for all i, j. In the second case no pair is matchable with element n, since $a_i + a_j \leq a_1 + a_2$ for all i, j. □

Property 2 *If $a_1 + a_{n-1} + a_n = b$ or $a_1 + a_2 + a_n = b$, then the instance can be reduced by assigning to a subset the three elements satisfying the condition.*

Proof. In the first case any pair (i, j) matchable with element 1 will have $a_i = a_{n-1}$ and $a_j = a_n$ (assuming $i < j$). In the second case any pair (i, j) matchable with element n will have $a_i = a_1$ and $a_j = a_2$ (assuming $i < j$). □

Property 3 *Let*

$$r = \max\{j : a_{j-1} + a_j > b - a_n\} \tag{4}$$

If $r > m$ then the instance has solution no.

Proof. By definition of r, any two elements of $\{1, \ldots, r\}$ are too large to be assigned to the same subset, so at least r subsets would be needed. □

If $r = m$, we know that each of the m subsets can be initialized to contain one of the first m elements. The residual problem, however, remains strongly NP-complete, since it is equivalent to *Numerical Matching with Target Sums* (problem [SP17] in Garey and Johnson [5]).

When Property 3 fails because (4) produces a value $r \leq m$, the value of r can be increased, in an attempt to possibly detect that the instance has solution *no*. Let us initialize a set I to $\{1, \ldots, r\}$. For $k = r + 1, \ldots, n$, let $e(k) = \min\{j : a_j + a_k \leq b - a_n\}$ (and observe that $e(k) < r$). No element of $\{1, \ldots, e(k) - 1\}$ can be assigned to the subset containing k. If for each $l \in \{i \in I : i \geq e(k)\}$, there is no $j \in N \setminus (I \cup \{k\})$ satisfying $a_k + a_l + a_j = b$, then we know that k cannot be assigned to any subset containing an element from I: hence $\{k\}$ can be added to I. Whenever this occurs, the test of Property 3 can be repeated with $r = |I|$.

Property 3 and the above improvement are based on "large" elements. Similar properties hold for "small" elements:

Property 4 *Let*

$$s = \min\{j : a_j + a_{j+1} < b - a_1\} \tag{5}$$

If $n - s + 1 > m$ *then the instance has solution* no.

Proof. By definition of s, any two elements of $\{s, \ldots, n\}$ are too small to be assigned to the same subset. □

In this case too, when the property fails (because (5) produces a value s such that $n - s + 1 \leq m$), we can make a further attempt. We initialize $I = \{s, \ldots, n\}$ and, for $k = s-1, s-2, \ldots, 1$, we determine $\tilde{e}(k) = \max\{j : a_j + a_k \geq b - a_1\}$: k is added to I if there is no pair (l, j) ($l \in \{i \in I : i \leq \tilde{e}(k)\}$, $j \in N \setminus (I \cup \{k\})$) for which $a_k + a_l + a_j = b$. At any iteration, if $|I| > m$ then we know that the instance has solution *no*.

Property 5 *Let*

$$\begin{aligned} t &= \max\{j : a_{j-2} + a_{j-1} + a_j > b\} && (6) \\ t' &= \min\{j : a_j + a_{j+1} + a_{j+2} < b\} && (7) \end{aligned}$$

If $\max\{\lceil t/2 \rceil, \lceil (n - t' + 1)/2 \rceil\} > m$ *then the instance has solution* no.

Proof. By definitions (6) and (7), no three elements of $\{1, \ldots, t\}$, nor three elements of $\{t', \ldots, n\}$, may be assigned to the same subset. □

Once the elements have been sorted according to (3), Properties 1 and 2 may be tested in constant time, while Properties 3–5 need linear time.

The improvements of Properties 3 and 4 may be implemented so as to require $O(n^2)$ time. Indeed, for each k: (a) $e(k)$ is determined in linear time, and (b) all required tests may be performed in linear time as follows. We check $a_k + a_l + a_j = b$ by increasing $l \geq e(k)$ and, for each l, by decreasing j, halting the search as soon as a j^* is found for which $a_k + a_l + a_{j^*} \geq b$: due to sorting, the search for $l + 1$ (if needed) can then start from $j = j^*$, since $a_k + a_{l+1} + a_j < b$ for $j = j^* + 1, \ldots, n$. Similar considerations hold for $\tilde{e}(k)$.

3.2 Advanced properties

When the properties of the previous section fail to solve the instance, stronger properties can be used, for which it is convenient to first generate all feasible element pairs.

In order to use a more compact notation, let us introduce an alternative representation of an instance of 3-PARTITION: let $W = \{w_1, \ldots, w_q\}$ be the set containing the different sizes of the elements of N, and let n_j $(j = 1, \ldots, q)$ be the number of elements having size w_j (i.e., $\sum_{j=1}^{q} n_j = n$). We assume that

$$w_1 > w_2 > \ldots > w_q \tag{8}$$

Given two sizes $w_j, w_k \in W$ (with j not necessarily different from k), we use a compact notation for representing the set of all pairs formed by one element of size w_j and one of size w_k: a triple $\pi_i = \langle \alpha(i), \beta(i), \nu(i) \rangle$ will represent such a set, where $w_{\alpha(i)}$ and $w_{\beta(i)}$ are the sizes and $\nu(i)$ is the number of pairs, given by

$$\nu(i) = \begin{cases} \min\{n_{\alpha(i)}, n_{\beta(i)}\} & \text{if } \alpha(i) \neq \beta(i) \\ \lfloor n_{\alpha(i)}/2 \rfloor & \text{otherwise} \end{cases} \tag{9}$$

We further denote by $\sigma(i)$ the size corresponding to triple π_i, i.e., $\sigma(i) = w_{\alpha(i)} + w_{\beta(i)}$. Let $\Pi = \{\pi_1, \ldots, \pi_p\}$ be the set of all triples.

Given a size w_j, let $M(j)$ be the set of the indices of those triples that represent pairs matchable with w_j, i.e.,

$$M(j) = \{i : \sigma(i) = b - w_j\} \tag{10}$$

Trivially, if a w_j exists such that $M(j) = \emptyset$ then the solution is *no*. Other situations in which either the solution is *no* or the instance can be reduced are detected by the following properties. We consider four mutually exclusive cases.

Given a set $M(j)$, observe that, from (10), we have, for any $i \in M(j)$: (a) $w_{\alpha(i)} \neq w_{\alpha(k)}$ and $w_{\alpha(i)} \neq w_{\beta(k)}$ for all $k \in M(j) \setminus \{i\}$; (b) $w_{\beta(i)} \neq w_{\alpha(k)}$ and $w_{\beta(i)} \neq w_{\beta(k)}$ for all $k \in M(j) \setminus \{i\}$. In other words, the pairs corresponding to the triples of $M(j)$ are obtained from $2\sum_{i \in M(j)} \nu(i)$ distinct elements.

Property 6 *Given a size w_j such that $w_j \neq w_{\alpha(i)}$ and $w_j \neq w_{\beta(i)}$ for each $i \in M(j)$,*

(i) *if $n_j > \sum_{i \in M(j)} \nu_i$ then the solution is* no;

(ii) *if* $n_j = \sum_{i \in M(j)} \nu_i$ *then the instance can be reduced by defining* n_j *subsets, each containing one element of size* w_j *and one pair corresponding to a triple* π_i *with* $i \in M(j)$.

Proof. Both cases immediately follow from the observation that each element of size w_j must be assigned to a subset not containing another element of size w_j. □

Property 7 *Given a size* w_j *such that, for each* $i \in M(j)$, $w_{\alpha(i)} = w_j$ *or* $w_{\beta(i)} = w_j$ *but* $w_{\alpha(i)} \neq w_{\beta(i)}$ *(hence* $|M(j)| = 1$*), let* $M(j) = \{k\}$*:*

(i) *if* n_j *is odd then the solution is* no*;*

(ii) *if* n_j *is even and* $n_j/2 > \nu_k$ *then the solution is* no*;*

(iii) *otherwise the instance can be reduced by defining* $n_j/2$ *subsets, each containing one element of size* w_j *and one pair corresponding to a triple* π_k.

Proof. First observe that any pair of total size $b - w_j$ contains exactly one element of size w_j, so any feasible subset either contains exactly two elements of size w_j or no such element. It follows that in any feasible solution the elements of size w_j are assigned to $n_j/2$ identical subsets (each containing two elements of size w_j and one of size $b - 2w_j$). In cases (i) and (ii) such an assignment is impossible, while in case (iii) (n_j even and $n_j/2 \leq \nu_k$) all such assignments are equivalent. □

Property 8 *Given a size* w_j *such that* $w_j = w_{\alpha(i)} = w_{\beta(i)}$ *for at least one* $i \in M(j)$, *let* $\overline{M} = \{i \in M(j) : w_{\alpha(i)} \neq w_j\}$. *Then*

(i) *if* $n_j \bmod 3 > \sum_{i \in \overline{M}} \nu(i)$ *then the solution is* no*;*

(ii) *if* $n_j \bmod 3 = \sum_{i \in \overline{M}} \nu(i)$ *then the instance can be reduced by defining* $\lfloor n_j/3 \rfloor$ *subsets, each containing three elements of size* w_j, *plus* $n_j \bmod 3$ *subsets, each containing one element of size* w_j *and one pair corresponding to a triple* π_i *with* $i \in \overline{M}$.

Proof. First observe that $w_j = b/3$, so no feasible subset can contain exactly two elements of size w_j. In case (i), even by defining the maximum possible number of subsets containing three elements of size w_j each, one or two such elements cannot be feasibly assigned. In case (ii) any feasible assignment of the elements of size w_j is equivalent to that in the thesis. □

Property 9 *Given a size* w_j, *let* $\widetilde{M} = \{i \in M(j) : w_{\alpha(i)} = w_j$ *or* $w_{\beta(i)} = w_j$ *but* $w_{\alpha(i)} \neq w_{\beta(i)}\}$, *and observe that* $|\widetilde{M}| \leq 1$. *If* $|\widetilde{M}| = 1$, *let* $\widetilde{M} = \{k\}$ *and* $\widehat{m} = \sum_{i \in M(j) \setminus \{k\}} \nu(i)$*: then, if* $\widehat{m} > 0$ *we have*

(i) *if* $n_j - 2\min\{\lfloor n_j/2\rfloor, \nu(k)\} > \widehat{m}$ *then the solution is* no;

(ii) *if* $n_j - 2\min\{\lfloor n_j/2\rfloor, \nu(k)\} = \widehat{m}$ *then the instance can be reduced by defining* $\min\{\lfloor n_j/2\rfloor, \nu(k)\}$ *subsets, each containing one element of size* w_j *and one pair corresponding to triple* π_k*, plus* $\widehat{m}$ *subsets, each containing one element of size* w_j *and one pair corresponding to a triple* π_i *with* $i \in M(j) \setminus \{k\}$*;*

(iii) *if* $n_j - 2\min\{\lfloor n_j/2\rfloor, \nu(k)\} < \widehat{m}$ *and* $\widehat{m} = 1$ *then the instance can be reduced by defining* $\min\{\lfloor n_j/2\rfloor, \nu(k)\}$ *subsets, each containing one element of size* w_j *and one pair corresponding to triple* π_k*.*

Proof. First observe that $w_j \neq b/3$, hence any feasible subset containing elements of size w_j will contain either one or two such elements. In case (i), even by defining the maximum possible number of subsets containing two elements of size w_j, the remaining elements of size w_j cannot be feasibly assigned. In case (ii) any feasible assignment of the elements of size w_j is equivalent to that in the thesis. The hypothesis in case (iii) implies that $n_j = 2\min\{\lfloor n_j/2\rfloor, \nu(k)\}$, hence n_j is even and the $\nu(k)$ pairs are sufficient for producing $n_j/2$ feasible subsets, each containing two elements of size w_j. If, in addition, $\widehat{m} = 1$, this is the only feasible way of assigning the elements of size w_j: indeed, if one such element were matched with the unique pair corresponding to the unique triple of $M(j) \setminus \{k\}$, then an odd number of elements of size w_j would remain, impossible to be feasibly assigned. □

We conclude this section by summarizing the four properties above, and showing that they imply an almost complete enumeration of the possible cases arising by relating a size w_j to the existence of the same size in the pairs matchable with w_j. To this end, let P_j be the set of size pairs matchable with w_j, i.e., $P_j = \{(w_{\alpha(i)}, w_{\beta(i)}) : i \in M(j)\}$. The decision-tree in Figure 1 enumerates all possibilities for a given w_j: the numbered nodes refer to the corresponding properties, and their descendant branches represent the possible decisions. In ten out of thirteen cases the instance is solved or reduced (‘*no*’ or ‘*red.*’ in the figure), while in the three remaining cases no decision can be taken (‘?’ in the figure).

4 Implementation

In this section we show how the properties of the previous sections can be efficiently implemented so as to obtain a reduction algorithm for 3-PARTITION.

The simple properties in Sections 3.1 can be immediately checked, once the elements have been sorted, while for the advanced properties in Section 3.2 it is convenient to use

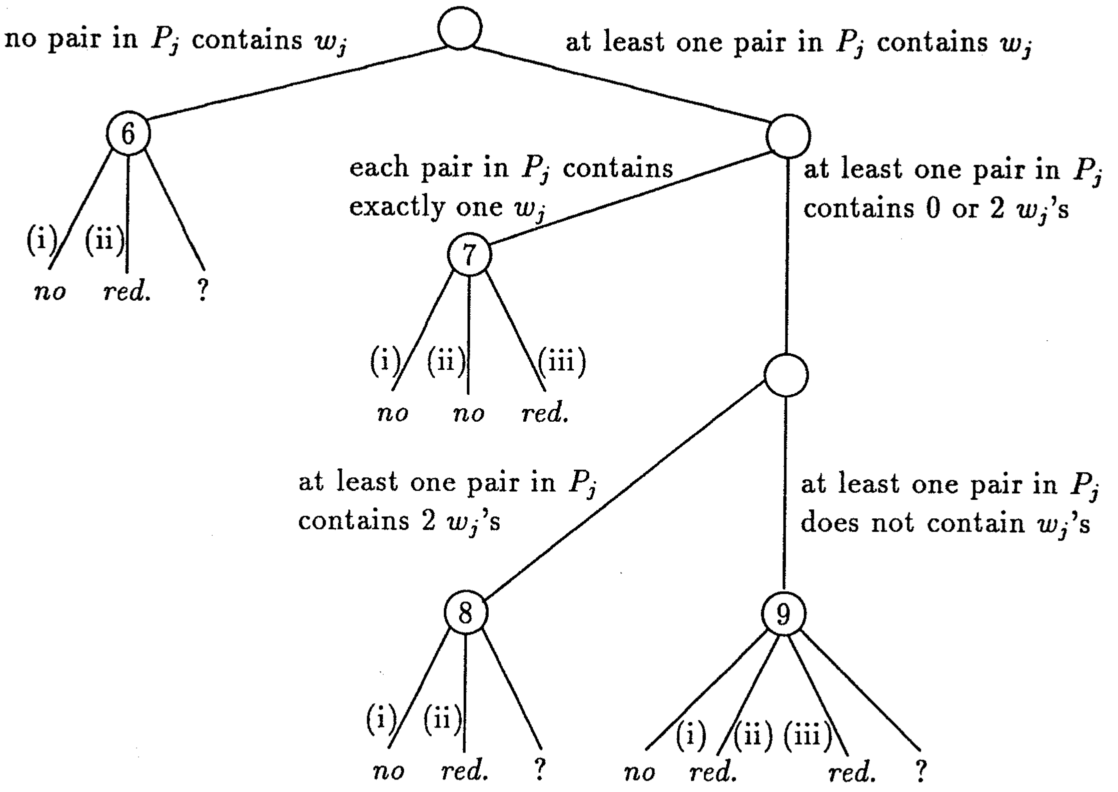

Figure 1: Decision-tree summarizing Properties 6–9.

a data structure that allows a fast check of the conditions as well as an efficient updating when reductions occur. A possible data structure is described in the following.

4.1 Data structure

We refer to the notation introduced in Section 3.2. Arrays $[w_1 \ldots w_q]$ and $[n_1 \ldots n_q]$, sorted according to (8), can be obtained in $O(n)$ time from the sorted values a_j. It is then easy to generate the p pairs in $O(q^2)$ time and to store the corresponding values into arrays α, β, ν and σ. While generating, in $O(q)$ time, all pairs including a given size w_j, it is easy to check (through a technique similar to that described at the end of Section 3.1, hence without increasing the computational complexity), whether the current candidate pair has a total size $\sigma(i)$ such that no element size $w_l = b - \sigma(i)$ exists: if this is the case, such a useless pair is not stored. We then sort the resulting arrays, in $O(q^2 \log q)$ time, so that

$$\sigma(1) \leq \sigma(2) \leq \ldots \leq \sigma(p) \tag{11}$$

Observe that the pairs matchable with a given size w_j are stored in consecutive locations of the arrays. Hence sets $M(j) (j = 1, \ldots, q)$ can be handled through an array μ of $q+1$ pointers, such that μ_j is the index of the first triple representing a pair matchable with w_j. More formally, the information on all triples π_i such that $i \in M(j)$ are stored in $(\alpha(x) \ldots \alpha(y))$, $(\beta(x) \ldots \beta(y))$, $(\nu(x) \ldots \nu(y))$ and $(\sigma(x) \ldots \sigma(y))$, with $x = \mu_j$ and $y = \mu_{j+1} - 1$ $(\mu_{q+1} = p + 1)$.

With this structure, testing any of Properties 6–9 for a given size w_j requires $O(|M(j)|)$ time. Since the $M(j)$'s are disjoint, testing the four properties for all sizes w_j $(j = 1, \ldots, q)$ requires $O(q^2)$ time in total.

In order to iteratively apply the properties of the previous section, reducing the current instance whenever possible, we also need an efficient way of updating the data structure without paying the $O(q^2)$ time per reduction resulting from a straightforward implementation. Since each reduction implies the removal of three elements from the instance, we need to eliminate all pairs containing such elements. To this end we use a sparse matrix R having q columns: each non-empty entry $R(k, j)$ stores the pointer to the k-th triple corresponding to a pair containing at least one element of size w_j (hence the number of non-empty entries per column is bounded by q). A reduction can then be performed in $O(q)$ time by executing the following operations for each element to be removed. Let w_j be the element size:

```
$n_j := n_j - 1$;
for each pointer $R(k, j)$ of column $j$ do
    if $\alpha(R(k,j)) \neq \beta(R(k,j))$ then $\nu(R(k,j)) := \min\{n_{\alpha(R(k,j))}, n_{\beta(R(k,j))}\}$
    else $\nu(R(k,j)) := \lfloor n_j/2 \rfloor$.
```

Example 1 Let $n = 12$, $(a_j) = (45, 45, 44, 35, 35, 30, 30, 28, 28, 27, 27, 26)$, and $b = 100$. We have $q = 7$,

$$\begin{aligned} (w_j) &= (45,\ 44,\ 35,\ 30,\ 28,\ 27,\ 26) \\ (n_j) &= (\ 2,\ \ 1,\ \ 2,\ \ 2,\ \ 2,\ \ 2,\ \ 1) \end{aligned}$$

The data structure, sorted according to (11), is thus

$$\begin{array}{lcllllllllll}
(\alpha(i)) & = & (\,5, & 4, & 5, & 3, & 3, & 2, & 1, & 2, & 1, & 2) \\
(\beta(i)) & = & (\,6, & 7, & 5, & 4, & 3, & 7, & 6, & 5, & 5, & 4) \\
(\nu(i)) & = & (\,2, & 1, & 1, & 2, & 1, & 1, & 2, & 1, & 2, & 1) \\
(\sigma(i)) & = & (55, & 56, & 56, & 65, & 70, & 70, & 72, & 72, & 73, & 74) \\
(\mu_j) & = & (\,1, & 2, & 4, & 5, & 7, & 9, & 10, & 11) & &
\end{array}$$

and the pointer matrix is

$$R = \begin{bmatrix} 7 & 6 & 4 & 2 & 1 & 1 & 2 \\ 9 & 8 & 5 & 4 & 3 & 7 & 6 \\ & 10 & & 10 & 8 & & \\ & & & & 9 & & \end{bmatrix}$$

Property 6 applies for $w_7 = 26$, hence we can reduce the instance by defining a subset containing sizes 26, 30 and 44. Using matrix R, the structure is modified, obtaining:

$$\begin{array}{lcllllllllll}
(n_j) & = & (\,2, & 0, & 2, & 1, & 2, & 2, & 0) & & & \\
(\nu(i)) & = & (\,2, & 0, & 1, & 1, & 1, & 0, & 2, & 0, & 2, & 0)\ \square
\end{array}$$

4.2 Reduction algorithm

The data structure of the previous section has been used to implement a reduction procedure RED, which can be summarized as follows.

```
procedure RED
sort the elements according to (3) and check Properties 1–5;
if the instance has not been solved then
  begin
    construct the data structure for the reduced instance;
    repeat
      for $j$ := 1 to $q$ do
        check Properties 6–9 on size $w_j$
    until no new reduction occurred
  end.
```

(The statement "check Property x" implies that the current instance is possibly reduced and the execution is halted if all elements have been reduced or the property ensures that the solution is *no*.)

The algorithm was improved through two heuristic attempts, to be executed when procedure RED cannot solve the instance. The first heuristic consists of a simple greedy algorithm which tries to construct a feasible solution for the reduced instance. At each iteration the algorithm selects an unassigned element according to a given criterion (see below): if there are pairs matchable with the element, then a new subset is defined, containing the element and the first such pair; otherwise, the algorithm terminates with no solution. The algorithm is executed three times, in sequence. The first two executions use the following criteria, respectively, for the selection of an element of size w_l (all quantities refer to the currently unassigned elements):

(a) $l = \arg\min \left\{ \sum_{i \in M(j)} \nu(i)/n_j \right\}$, breaking ties arbitrarily;

(b) same as (a), but ties are broken by selecting the size that minimizes

$$\sum_{i \in M(l)} \left(\sum_{h \in M(\alpha(i))} \nu(h) + \sum_{h \in M(\beta(i))} \nu(h) \right)$$

The third execution uses criterion (b) and, in addition, performs reduction procedure RED at each iteration, i.e., whenever a new subset has been defined.

When the greedy attempts fail to find a feasible solution, a decision tree is partially explored, in an attempt to solve the current reduced instance. At each decision node, the next size w_j with $n_j > 0$ is selected and $\sum_{i \in M(j)} \nu(i)$ descendant nodes are generated by assigning to a subset an element of size w_j and a matchable pair. Procedure RED is executed at each node, for the current instance: if solution *yes* is obtained, the execution is obviously halted; if solution *no* is obtained, the node is fathomed; otherwise, the exploration proceeds. The tree is explored with depth-first strategy, until a prefixed limit L on the number of decision nodes is reached.

5 Computational Experiments

We have coded in FORTRAN 77 the algorithm of the previous section, called DM3 in the following, and executed a series of computational experiments on a Digital VAXstation 3100 (whose speed is roughly equal to two thirds that of a PC 486/33) with a limit $L = 20,000$ on the number of decision nodes explored in the final enumerative phase.

Four classes of randomly generated test problems have been considered. It is worth noting that generating random instances of 3-PARTITION is not trivial, due to the need to satisfy condition $\sum_{j=1}^{n} a_j = mb$. To our knowledge, no benchmarks or instance generators have been proposed so far in the literature. Given n, m and b, our test instances are as follows.

Class 1 is obtained by generating the element sizes in two phases. In the first phase sizes $a_1, a_2, \ldots, a_t$ are uniformly randomly generated in the interval $[1, b-2]$, until either $t = n$ or $\sum_{j=1}^{t} a_j \geq mb$. If $t = n$, the second phase executes $mb - \sum_{j=1}^{n} a_j$ iterations: at each iteration an a_j is randomly selected, and its value is increased by one unit. If instead $\sum_{j=1}^{t} a_j \geq mb$, a_t is set to $mb - \sum_{j=1}^{t-1} a_j$, and the second phase executes $n - t$ iterations: at each iteration an $a_j > 1$ is randomly selected, a value ϑ is randomly generated in the interval $[1, a_j - 1]$, a new element of size $a_j - \vartheta$ is defined, and the size of the jth element is decreased to ϑ.

Class 2 is obtained by first generating $m - 2$ groups of three elements such that the total size in each group equals b: the first size, say a_k, of a group is randomly generated in the interval $[1, b-2]$, the next one in $[1, b - a_k - 1]$, and the last one is set to $b - a_k - a_{k+1}$. Five additional elements are then generated with random sizes in the interval $[1, \lfloor \frac{2}{3} b \rfloor]$: if their total size, say Θ, is such that $b + 2 \leq \Theta \leq 2b - 1$, the last size is set to $2b - \Theta$, otherwise a new group of five elements is generated from scratch.

The next two classes have been defined so as to obtain instances satisfing condition (1).

Class 3 is defined through a linear probability density function, $f(x) = 16 - 32x$ in the interval $\frac{1}{4} < x < \frac{1}{2}$. The mean value of such distribution is $\frac{1}{3}$, so, by generating n values and multiplying each of them by b, the expected total value is mb, and few values need be modified in the second phase in order to satisfy $\sum_{j=1}^{n} a_j = mb$. A random value distributed according to $f(x)$ should be obtained from a uniform value r in the interval $(0, 1]$ through the transformation $x = \frac{1}{2} - \frac{1}{4}\sqrt{r}$. Remind however that we consider integer sizes, and assume, for simplicity, that b is divisible by four: hence in the first phase, for each random value $r \in (0, 1]$, we define a size $a_j = \lfloor \frac{b}{2} - 1 - \sqrt{r}(\frac{b}{4} - 2) \rfloor$. The resulting values are at least $\frac{b}{4} + 1$ and at most $\frac{b}{2} - 2$. ($a_j = \frac{b}{2} - 1$ would not be matchable with any pair.) If $\sum_{j=1}^{n} a_j > mb$ (resp. $< mb$), the second phase performs $|\sum_{j=1}^{n} a_j - mb|$ iterations: at each iteration an $a_j > \frac{b}{4} + 1$ (resp. $< \frac{b}{2} - 2$) is randomly selected and its value is decreased (resp. increased) by one unit.

Class 4 is obtained by first generating $m - 2$ groups of three elements in a way similar to that used for Class 2. For each group, the first size, say a_k, is randomly generated in the interval $[\frac{b}{4}+1, \frac{b}{2}-2]$, the next one in $[\frac{b}{4}+1, b-a_k-(\frac{b}{4}+1)]$, and the last one is set to $b-a_k-a_{k+1}$. The six last elements are then generated as in Class 3.

For each class we have considered different values of n divisible by three ($n = 24$, 51, 99, 249, 501, 999) and two values of b (100 and 1000). For each triple (*class*,n,b), ten instances have been generated.

In a first series of experiments we have compared, on small-size instances, DM3 with the three approaches described in Section 2. The transformed BPP, and MKP instances have been solved, respectively, with FORTRAN codes MTP and MTM (from the diskette accompanying the book by Martello and Toth [8]), both with a limit of 200,000 iterations. The transformed $P||C_{\max}$ instances have been solved with the C language implementation of the algorithm of Dell'Amico and Martello [1], with a limit of 20,000 iterations. The results in Table 1 refer to instances with $n < 100$ and $b = 100$: the entries give the average computing time and the number of cases in which the solution was determined, over ten random instances. The results clearly show that algorithms from the literature have a very poor performance when applied to 3-PARTITION instances, while our specialized procedure was able to solve all the instances in less than two seconds, on average. Hence, the following experiments were performed only on DM3.

Table 1: VAXstation 3100 seconds; $b = 100$;
average times and number of solved instances over ten.

		BPP		$P\|\|C_{\max}$		MKP		DM3	
Class	n	time	# sol.	time	# sol.	time	# sol.	time	# sol.
1	24	0.11	10	7.76	10	881.91	1	0.01	10
	51	246.33	8	133.40	1	770.22	0	0.36	10
	99	1067.56	0	241.78	0	1445.50	0	1.14	10
2	24	0.22	10	1.42	10	360.43	2	0.03	10
	51	288.92	6	97.92	4	695.66	0	0.44	10
	99	967.83	0	239.07	0	527.57	0	1.17	10
3	24	0.56	10	0.61	10	160.94	3	0.02	10
	51	323.97	5	43.09	7	315.73	2	0.09	10
	99	897.38	0	204.37	4	354.29	0	0.09	10
4	24	0.54	10	0.38	10	79.85	7	0.03	10
	51	270.08	5	19.76	10	170.01	4	0.06	10
	99	965.02	0	192.53	5	369.69	1	0.11	10

The results in Table 2 refer to instances with n up to 999, and to cases $b = 100$ and $b = 1000$. The entries give the same information as in Table 1, plus the average number of iterations in the enumerative phase, and the number of solved instances having solution *yes*. The average CPU time and number of iterations include the effort spent on the unresolved instances, for which the prefixed limit was reached in the enumerative phase. Procedure DM3 solved quite easily all instances with $b = 100$. The instances with $b = 1000$ are clearly more difficult: the properties in Section 3 tend indeed to be less effective if the number of different triples is high. The overall behaviour of DM3 is satisfactory: we could solve 470 instances out of 480 within reasonable CPU times.

Table 2: procedure DM3; VAXstation 3100 seconds;
average times and iterations, and number of solved instances over ten, number of *yes*.

		$b = 100$				$b = 1000$			
Class	n	time	iter.	# sol.	# *yes*	time	iter.	# sol.	# *yes*
	24	0.01	0	10	0	0.01	0	10	0
	51	0.36	8	10	3	0.01	0	10	0
1	99	1.14	11	10	9	0.68	0	10	0
	249	3.30	9	10	10	457.46	10333	8	8
	501	13.89	380	10	10	442.62	3137	9	9
	999	14.54	0	10	10	1847.97	4313	9	9
	24	0.03	0	10	2	0.01	0	10	0
	51	0.44	8	10	10	0.01	0	10	0
2	99	1.17	5	10	10	2.83	2	10	0
	249	2.70	0	10	10	414.90	9363	8	8
	501	5.94	0	10	10	562.38	3165	9	9
	999	19.55	254	10	10	1565.22	3362	9	9
	24	0.02	0	10	3	0.01	0	10	0
	51	0.09	2	10	9	0.34	2	10	1
3	99	0.09	0	10	10	1.59	3	10	6
	249	0.65	0	10	10	8.65	0	10	8
	501	0.59	0	10	10	30.19	48	10	8
	999	0.84	0	10	10	58.13	31	10	9
	24	0.03	0	10	10	0.01	0	10	0
	51	0.06	0	10	10	0.46	2	10	3
4	99	0.11	2	10	10	5.48	189	10	10
	249	0.24	0	10	10	12.99	46	10	10
	501	0.56	0	10	10	187.48	3619	9	9
	999	1.28	0	10	10	306.23	2199	9	9

A final observation concerns the number of instances having solution *yes*. It clearly increases with n, as the number of feasible groups of three elements grows steeply. The number of *yes* decreases with b, since the range of possible total sizes of a group of three elements increases, hence there is a lower probability that it is equal to b.

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References

[1] M. Dell'Amico, S. Martello, "Optimal Scheduling of Tasks on Identical Parallel Processors", *ORSA Journal on Computing* 7, 191–200, 1995.

[2] M.R. Garey, D.S. Johnson, "Complexity Results for Multiprocessors Scheduling Under Resource Constraints", *SIAM Journal on Computing* 4, 397–411, 1975.

[3] M.R. Garey, D.S. Johnson, R. Sethi, "The Complexity of Flowshop and Jobshop Scheduling", *Mathematics of Operations Research* 1, 117–129, 1976.

[4] M.R. Garey, D.S. Johnson, " "Strong" NP-Completeness Results: Motivation, Examples and Implications", *Journal of ACM* 25, 499–508, 1978.

[5] M.R. Garey, D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, San Francisco, 1979.

[6] J.A. Hoogeveen, J.K. Lenstra, S.L. van de Velde, "Sequencing and Scheduling", in M. Dell'Amico, F. Maffioli, S. Martello (eds.), *Annotated Bibliographies in Combinatorial Optimization*, Wiley, Chichester, 1997.

[7] J.K. Lenstra, A.H.G. Rinnooy Kan, P. Brucker, "Complexity of Machine Scheduling Problems", *Annals of Discrete Mathematics* 1, 343–362, 1977.

[8] S. Martello, P. Toth, *Knapsack Problems: Algorithms and Computer Implementations*, Wiley, Chichester, 1990.

1. Maria Cristina Marcuzzo [1985] "Yoan Violet Robinson (1903-1983)", pp. 134
2. Sergio Lugaresi [1986] "Le imposte nelle teorie del sovrappiù", pp. 26
3. Massimo D'Angelillo e Leonardo Paggi [1986] "PCI e socialdemocrazie europee. Quale riformismo?", pp. 158
4. Gian Paolo Caselli e Gabriele Pastrello [1986] "Un suggerimento hobsoniano su terziario ed occupazione: il caso degli Stati Uniti 1960/1983", pp. 52
5. Paolo Bosi e Paolo Silvestri [1986] "La distribuzione per aree disciplinari dei fondi destinati ai Dipartimenti, Istituti e Centri dell'Università di Modena: una proposta di riforma", pp. 25
6. Marco Lippi [1986] "Aggregations and Dynamic in One-Equation Econometric Models", pp. 64
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8. Mario Forni [1986] "Storie familiari e storie di proprietà. Itinerari sociali nell'agricoltura italiana del dopoguerra", pp. 165
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10. Nerio Naldi [1986] "L'efficienza marginale del capitale nel breve periodo", pp. 54
11. Fernando Vianello [1986] "Labour Theory of Value", pp. 31
12. Piero Ganugi [1986] "Risparmio forzato e politica monetaria negli economisti italiani tra le due guerre", pp. 40
13. Maria Cristina Marcuzzo e Annalisa Rosselli [1986] "The Theory of the Gold Standard and Ricardo's Standard Comodity", pp. 30
14. Giovanni Solinas [1986] "Mercati del lavoro locali e carriere di lavoro giovanili", pp. 66
15. Giovanni Bonifati [1986] "Saggio dell'interesse e domanda effettiva. Osservazioni sul cap. 17 della General Theory", pp. 42
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18. Mario Forni [1986] "Aggregazione ed esogeneità", pp. 13
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20. Fiorenzo Sperotto [1987] "L'immagine neopopulista di mercato debole nel primo dibattito sovietico sulla pianificazione", pp. 34
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24. Fernando Vianello [1987] "Effective Demand and the Rate of Profits. Some Thoughts on Marx, Kalecki and Sraffa", pp. 41
25. Anna Maria Sala [1987] "Banche e territorio. Approccio ad un tema geografico-economico", pp. 40
26. Enzo Mingione e Giovanni Mottura [1987] "Fattori di trasformazione e nuovi profili sociali nell'agricoltura italiana: qualche elemento di discussione", pp. 36
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28. Massimo Matteuzzi e Annamaria Simonazzi [1988] "Il debito pubblico", pp. 62
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30. Paolo Bosi [1988] "MICROMOD. Un modello dell'economia italiana per la didattica della politica fiscale", pp. 34
31. Paolo Bosi [1988] "Indicatori della politica fiscale. Una rassegna e un confronto con l'aiuto di MICROMOD", pp. 25
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38. Giovanni Mottura [1988] "La 'persistenza' secolare. Appunti su agricoltura contadina ed agricoltura familiare nelle società industriali", pp. 40
39. Giovanni Mottura [1988] "L'anticamera dell'esodo. I contadini italiani della 'restaurazione contrattuale' fascista alla riforma fondiaria", pp. 40
40. Leonardo Paggi [1988] "Americanismo e riformismo. La socialdemocrazia europea nell'economia mondiale aperta", pp. 120
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46. Mario Forni [1989] "Trend, Cycle and 'Fortuitous cancellation': a Note on a Paper by Nelson and Plosser", pp. 4
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48. Roberto Golinelli [1989] "Note sulla struttura e sull'impiego dei modelli macroeconometrici", pp. 21
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50. Gian Paolo Caselli e Gabriele Pastrello [1989] "The Linkage between Tertiary and Industrial Sector in the Italian Economy: 1951-1988. From an External Dependence to an International One", pp. 40
51. Gabriele Pastrello [1989] "Francois quesnay: dal Tableau Zig-zag al Tableau Formule: una ricostruzione", pp. 48
52. Paolo Silvestri [1989] "Il bilancio dello stato", pp. 34
53. Tim Mason [1990] "Tre seminari di storia sociale contemporanea", pp. 26
54. Michele Lalla [1990] "The Aggregate Escape Rate Analysed throught the Queueing Model", pp. 23
55. Paolo Silvestri [1990] "Sull'autonomia finanziaria dell'università", pp. 11

56. Paola Bertolini, Enrico Giovannetti [1990] "Uno studio di 'filiera' nell'agroindustria. Il caso del Parmigiano Reggiano", pp. 164

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58. Michele Lalla [1990] "Modelling Employment Spells from Emilia Labour Force Data", pp. 18

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60. Andrea Giommi [1990] "La probabilità individuale di risposta nel trattamento dei dati mancanti", pp. 13

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63. Andrea Ginzburg [1990] "Debito pubblico, teorie monetarie e tradizione civica nell'Inghilterra del Settecento", pp. 30

64. Mario Forni [1990] "Incertezza, informazione e mercati assicurativi: una rassegna", pp. 37

65. Mario Forni [1990] "Misspecification in Dynamic Models", pp. 19

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71. Margherita Russo [1990] "Distretti industriali in teoria e in pratica: una raccolta di saggi", pp. 119

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75. Enrico Giovannetti [1990] "Crisi e mercato del lavoro in un distretto industriale: il bacino delle ceramiche. Sez I", pp. 150

76. Enrico Giovannetti [1990] " Crisi e mercato del lavoro in un distretto industriale: il bacino delle ceramiche. Sez. II", pp. 145

78. Antonietta Bassetti e Costanza Torricelli [1990] "Una riqualificazione dell'approccio bargaining alla selezioni di portafoglio", pp. 4

77. Antonietta Bassetti e Costanza Torricelli [1990] "Il portafoglio ottimo come soluzione di un gioco bargaining", pp. 15

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93. Gian Paolo Caselli e Luca Martinelli [1993] "Italian GPN growth 1890-1992: a unit root or segmented trend representatin?", pp. 30

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95. Alberto Rinaldi [1993] "Lo sviluppo dell'industria metalmeccanica in provincia di Modena: 1945-1990", pp. 70

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