\\ 244 \\

The generalized dynamic factor model:
identification and estimation

by

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ABSTRACT

This paper analyzes identification conditions, and proposes an estimator, for a dynamic factor model where the idiosyncratic components are allowed to be mutually non-orthogonal. This model, which we call *generalized dynamic factor model*, is novel to the literature, and generalizes the static approximate factor model of Chamberlain and Rothschild (1983), as well as the exact factor model *à la* Sargent and Sims (1977). We prove mean-square convergence of our estimator to the common factor as the time cross-sectional dimensions go to infinity at appropriate rates. Simulations yield encouraging results in small samples. An empirical example on the output growth of US states illustrates the method.

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1. Introduction[1]

Dynamic factor models have been used extensively in finance for the analysis of asset prices, and in macroeconomics to study the business cycle. The assumption underlying these models is that the dynamics of multivariate time series can be modeled as the effect of a small number of common factors and of idiosyncratic components. This approach is particularly useful when the dimension of the system to be analysed is large, since it provides a parsimonious dynamic representation, whereas traditional VARMA models would require the estimation of too many parameters. In fact, many problems in economics and finance require the dynamic analysis of large number of assets and many sectoral, regional or individual variables. In economics in particular, the researcher has to face data sets which typically contain many cross-sectional information (n large) and a relatively short time period (T small). In such context, the use of factor models seems to be particularly appropriate, as shown by a small but growing recent literature in macroeconomics (Quah and Sargent, 1993, Forni and Reichlin, 1996, 1997, 1998, Forni an Lippi, 1997, Stock and Watson, 1998).

In this paper, as in Forni and Lippi (1998), we propose a very general factor model, which is novel to the literature. Since we allow the idiosyncratic components to be cross-correlated, individual shocks may have dynamic effects to other sectors. This is a more realistic assumption than the traditional orthogonality assumption underlying the traditional exact factor representation; it is particularly interesting for the analysis of that class of business cycle models in which local shocks propagate throughout the economy because of spillovers and complementarities (e.g., Cooper and Haltiwanger, 1990 and Davis and Haltiwanger, 1992), or because of input-output relations (e.g., Long and Plosser, 1982). The non-orthogonal case is commonly considered in the financial literature, where, however, it is restricted to static models. Our model encompasses, as a special case, the static approximate factor model of Chamberlain (1983) and Chamberlain and Rothschild (1983), and generalizes the dynamic factor model of Sargent and Sims (1977) and Geweke (1977) to the case of non-orthogonal idiosyncratic components.

Our representation differs from Stock and Watson (1998), where factors (apart from time-varying coefficients) are static, but growing in number with the cross-sectional dimension. An important feature of our model is that the common component is allowed to have an infinite Moving Average (MA) representation, so as to accommodate for both autoregressive (AR) and MA factors. It is more general than the finite dynamic factor model, which can be analyzed as a static factor model where lagged factors are treated as additional static factors. The infinite dimensional dynamic model is a relevant generalization, since AR factors are likely to arise in

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macroeconomic data with business cycle features.

We provide both identification and estimation results. We show that, although the model for finite n is not identified in general, under certain conditions, it is identified for n tending to infinity. We provide these conditions and we use them as a basis for a heuristic method for the identification of the number of common factors.

For estimation, we propose a method which works well in situations where the cross-sectional dimension is large and traditional estimation methods, based on maximum likelihood, are not appropriate. The basic idea of the estimator as an aggregate is a development of Forni and Reichlin (1998), who show uniform consistency for an estimator constructed from cross-sectional averages. The estimator proposed here is based on principal components, i.e., on a weighted average of observations. We show that the projection of the variables onto the leads and lags of the dynamic principal components converges to the common factor space for both n and T going to infinity.

In the static case, a principal component estimator has been used by Connor and Korajczyk (1986), who build on results in Chamberlain and Rothschild (1983) to show convergence for n going to infinity and T fixed. Stock and Watson (1998) use the same estimator for a more general model and provide consistency results for n and T going to infinity at some rate.

This paper is closely related to Forni and Lippi (1998), who develop the representation theory for the same model we discuss in this paper.

2. The Model

All stochastic variables under study are members of the standard Hilbert space $L_2(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is some given probability space. We will consider a double sequence $\mathbf{x} = \{x_{it},\ i \in \mathrm{N};\ t \in \mathrm{Z}\}$, whose model is

$$x_{it} = b_{i1}(L)u_{1t} + b_{i2}(L)u_{2t} + \cdots + b_{iq}(L)u_{qt} + \xi_{it}, \tag{1}$$

where the following Assumptions 1-4 are made.

ASSUMPTION 1.
(I) The q-dimensional vector process $\{(u_{1t},\ u_{2t},\ \cdots,\ u_{qt})',\ t \in \mathrm{Z}\}$ is orthonormal white noise, i.e. $\mathrm{var}(u_{jt}) = 1$ for any j and t, $u_{jt} \perp u_{jt-k}$ for any j, t and $k \neq 0$, $u_{jt} \perp u_{st-k}$ for any $s \neq j$, t and k;
(II) $\boldsymbol{\xi} = \{\xi_{it},\ i \in \mathrm{N},\ t \in \mathrm{Z}\}$ is a double sequence such that, firstly,

$$\boldsymbol{\xi}_n = \{(\xi_{1t},\ \xi_{2t},\ \ldots,\ \xi_{nt})',\ t \in \mathrm{Z}\}$$

is a wide-sense stationary vector process for any n, and, secondly, $\xi_{it} \perp u_{jt-k}$ for any i, j, t and k;
(III) the filters $b_{ij}(L)$ are square summable and bilateral.

Model (1) is a factor analytic model. The variables u_{jt} and $\chi_{it} = x_{it} - \xi_{it}$, $j = 1, \ldots, q$, will be called the *common factors* and the *common component* of x_{it}, respectively. The variable ξ_{it} will be called the *idiosyncratic component* of x_{it}.

Note that Assumption 1 implies that the n-dimensional vector process

$$\mathbf{x}_n = \{(x_{1t},\ x_{2t},\ \ldots,\ x_{nt})',\ t \in \mathrm{Z}\}$$

is stationary for any n. Note also that the filters $b_{ij}(L)$ may contain negative powers of the lag operator L, so that in general $b_{ij}(L)u_{jt}$ contains both lags and leads of u_{jt}. Clearly if (1) is interpreted as a structural representation, where the common factors have an economic meaning, then it would be appropriate to assume unilateral impulse-response functions. Since in this paper we are not concerned with the structural interpretation, we allow for general bilater filters.

The following restriction, though not strictly necessary, will considerably simplify the proofs of our results.

ASSUMPTION 2. Both the process $\boldsymbol{\xi}_n$ and $\boldsymbol{\chi}_n = \{(\chi_{1t},\ \chi_{2t},\ \ldots,\ \chi_{nt})',\ t \in \mathrm{Z}\}$ have rational spectral density for any n (thus, the filters $b_{ij}(L)$ are rational).

Obviously, Assumption 2 implies that the spectral density of $\mathbf{x}_n$ is rational for any n, and therefore defined everywhere and continuous on $[-\pi,\ \pi]$.

The main features of the model are the following. First, it is dynamic as in Geweke (1977) and Sargent and Sims (1977). Second, in contrast with the traditional dynamic factor model, the cross-sectional dimension is infinite. This feature is the same as in the static factor model of Chamberlain (1983) and Chamberlain and Rothschild (1983). An infinite cross-section, together with Assumptions 3 and 4 below, is crucial for identification of our model: indeed, and this is the third distinctive feature of (1), which differentiates it from the traditional dynamic factor model, we are not assuming mutual orthogonality of the idiosyncratic components ξ_{it}. Without orthogonality, for fixed n, reasonable assumptions allowing for identification of the idiosyncratic and the common component would be very hard to find.

Let $\mathbf{\Sigma}_n^x$ be the spectral density matrix of $\mathbf{x}_n$, and denote by λ_{nj}^x the function associating with any $\theta \in [-\pi, \pi]$ the real non-negative j-th eigenvalue of $\mathbf{\Sigma}_n^x(\theta)$ in descending order of magnitude. The functions λ_{nj}^x will be called the *dynamic eigenvalues* of $\mathbf{\Sigma}_n^x$.[2] In the same way, using obvious notation, λ_{nj}^χ and λ_{nj}^ξ denote the dynamic eigenvalues of $\mathbf{\Sigma}_n^\chi$ and $\mathbf{\Sigma}_n^\xi$, respectively. The latter will be called common and idiosyncratic eigenvalues respectively.

ASSUMPTION 3. The first idiosyncratic dynamic eigenvalue λ_{n1}^ξ is uniformly bounded, i.e., there exists a real Λ such that $\lambda_{n1}^\xi(\theta) \leq \Lambda$ for any $\theta \in [-\pi, \pi]$ and any n.

ASSUMPTION 4. The first q common dynamic eigenvalues diverge almost everywhere in $[-\pi, \pi]$, i.e., $\lim_{n\to\infty} \lambda_{nj}^\chi(\theta) = \infty$ for $j \leq q$, a.e. in $[-\pi, \pi]$.

Assumption 3 is clearly satisfied if the x's are mutually orthogonal at any lead and lag and have uniformly bounded spectral densities, but is more general as it allows, so to speak, for a limited amount of dynamic cross-correlation. Similarly, Assumption 4 guarantees a minimum amount of cross-correlation between the common components. With a slight oversimplification, Assumption 4 implies that each u_j is present in infinitely many cross-sectional units, with non-decreasing importance. On the contrary, Assumption 3 implies that idiosyncratic causes of variation, although possibly shared by many (even all) units, have their effect concentrated on a finite number of units and tending to zero as i tends to infinity.[3]

These assumptions have the following crucial consequence, the proof of which is given in the Appendix.

PROPOSITION 1. *Under Assumptions 1 through 4, the first q eigenvalues of $\mathbf{\Sigma}_n^x$*

[2] We use the terminology 'dynamic eigenvalues' to insist on the difference between the functions λ and the eigenvalues of the variance-covariance matrix employed in the static principal component analysis. A standard reference for eigenvalues and eigenvectors of the spectral density matrix is Brillinger (1981), Chapter 9.

[3] As a further illustration of Assumptions 3 and 4 see the example in Remark 2, Section 3 below.

diverge, as $n \to \infty$, a.e. in $[-\pi, \pi]$, whereas the $(q+1)$-th one is uniformly (with respect to θ) bounded.

The importance of Proposition 1 lies in the fact that it transforms statements on the dynamic eigenvalues of the unobservable $\boldsymbol{\chi}_n$ and $\boldsymbol{\xi}_n$ into statements on the dynamic eigenvalues of the $\mathbf{x}_n$. Thus, if the analysis of the dynamic eigenvalues of the observed process leads to the conclusion that the first q eigenvalues diverge a.e. in $[-\pi, \pi]$, whereas the $(q+1)$-th one is uniformly bounded, then the hypothesis of a model with q factors is plausible.

We call model (1), under Assumptions 1 to 4, the *generalized dynamic factor model* (GDFM).

We will show that, under Assumptions 1 through 4, the common components χ_{it} and the idiosyncratic components ξ_{it} are asymptotically identified and can be estimated. On the other hand, the identification and estimation of the common factors u_{jt} and the filters $b_{ij}(L)$, while being obviously of great interest when representation (1) is interpreted as structural, are beyond the scope of this paper.[4]

3. Averaging Sequences and Asymptotic Aggregates

In the next section, we prove that the common components χ_{it} can be obtained as limits of linear combinations of the observations. To get an idea of the linear combinations we would like to use, consider the simple example

$$x_{it} = u_t + \xi_{it}, \; i \in \mathrm{N}, \; t \in \mathrm{Z}, \tag{2}$$

where $\boldsymbol{\xi}_n$ is assumed to be orthonormal white noise. This implies that the spectral density matrix $\mathbf{\Sigma}_n^\xi$ is the $n \times n$ identity matrix, so that Assumption 3 is fulfilled. In the sequel, we will denote by $\mathbf{x}_{nt}$ the vector of variables $(x_{1t}, \; x_{2t}, \; \ldots, \; x_{nt})'$ (so that for the vector process $\mathbf{x}_n$ defined in the previous section, we have $\mathbf{x}_n = \{\mathbf{x}_{nt}, \; t \in \mathrm{Z}\}$). Analogous meanings are to be given to $\boldsymbol{\chi}_{nt}$ and $\boldsymbol{\xi}_{nt}$.

Consider the arithmetic mean of $\mathbf{x}_{nt}$:

$$\frac{1}{n}\sum_{i=1}^{n} x_{it} = u_t + \frac{1}{n}\sum_{i=1}^{n} \xi_{it}.$$

When n tends to infinity, the variance of the second term on the right hand side, which equals $1/n$, tends to zero, so that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_{it} = u_t$$

[4] On the identification and estimation of the common factors in a related model, see Forni and Reichlin (1998).

in the mean square. The same effect, asymptotic canceling of the idiosyncratic component, is obtained by taking any system of weighting vectors $\{\mathbf{a}_n,\ n \in \mathrm{N}\}$, with $\mathbf{a}_n = (a_{n1},\ a_{n2},\ \cdots,\ a_{nn})$, instead of the uniform weights employed for computing the standard average

$$\mathbf{w}_n = (1/n,\ 1/n,\ \cdots,\ 1/n),$$

provided that

$$\lim_{n\to\infty} |\mathbf{a}_n|^2 = \lim_{n\to\infty} \sum_{j=1}^{n} a_{nj}^2 = 0.$$

Going back to our general dynamic model, we not only will allow for general weights, but also for linear combinations involving leads and lags of the x's.

DEFINITION 1. A *dynamic averaging sequence*, DAS henceforth, is a sequence $\{\mathbf{a}_n(L),\ n \in \mathrm{N}\}$ where $\mathbf{a}_n(L)$ is the row vector

$$(a_{n1}(L),\ a_{n2}(L),\ \cdots,\ a_{nn}(L)),$$

$a_{ni}(L)$ being a square summable bilateral filter, with the condition

$$\lim_{n\to\infty} \sum_{i=1}^{n} \int_{-\pi}^{\pi} |a_{ni}(e^{-i\theta})|^2 d\theta = 0.$$

DEFINITION 2. Suppose that $\{\mathbf{a}_n(L),\ n \in \mathrm{N}\}$ is a DAS and that $\mathbf{a}_n(L)\mathbf{x}_{nt}$ converges in the mean square. We say that $y_t = \lim_{n\to\infty} \mathbf{a}_n(L)\mathbf{x}_{nt}$ is an *asymptotic aggregate* of the x's.

4. Recovering the Common Components

The following result shows that, by averaging with any DAS, the idiosyncratic component cancels asymptotically.

PROPOSITION 2. *If Assumptions 1, 2 and 3 are fulfilled, then*

$$\lim_{n\to\infty} \mathbf{a}_n(L)\boldsymbol{\xi}_{nt} = 0 \qquad (3)$$

in the mean square for any DAS $\{\mathbf{a}_n(L),\ n \in \mathrm{N}\}$.

PROOF. Denote by $\tilde{A}$ the transposed complex conjugate of a matrix A. Moreover, we recall that $\lambda_{n1}^{\xi}(\theta)$ is the maximum of $\mathbf{b}\boldsymbol{\Sigma}_n^{\xi}(\theta)\mathbf{b}'$ under the constraint $|\mathbf{b}| = 1$ (see

the proof of Lemma 1 in the Appendix for a more general result). Using Assumption 3, we obtain

$$\begin{aligned}||\mathbf{a}_n(L)\boldsymbol{\xi}_{nt}||^2 &= \int_{-\pi}^{\pi} \mathbf{a}_n(e^{-i\theta})\boldsymbol{\Sigma}_n^{\xi}(\theta)\tilde{\mathbf{a}}_n(e^{-i\theta})d\theta \\ &\le \int_{-\pi}^{\pi} \lambda_{n1}^{\xi}(\theta)\left(\sum_{i=1}^{n}|a_{ni}(e^{-i\theta})|^2\right)d\theta \le \Lambda \sum_{i=1}^{n}\int_{-\pi}^{\pi}|a_{ni}(e^{-i\theta})|^2 d\theta.\end{aligned}$$

QED

Proposition 2 generalizes to variables fulfilling Assumption 3 the statement that can be proved by elementary considerations when the components ξ_{it} are strictly idiosyncratic, i.e. mutually orthogonal at any lead and lag and of uniformly bounded spectral densities. Assumption 3 thus provides a good motivation for calling ξ_{it} the idiosyncratic component of x_{it}. As already noted in the Introduction, Assumption 3 allows for very interesting economic cases that lie between common and strictly idiosyncratic components. Forni and Lippi (1998) show that Assumption 3 is also necessary for (3) to hold for any DAS.

Given a subset $\mathcal{Y}$ of $L_2(\Omega, \mathcal{F}, P)$, let us denote by $\overline{\mathrm{span}}(\mathcal{Y})$ the minimum closed subspace of $L_2(\Omega, \mathcal{F}, P)$ containing $\mathcal{Y}$, and set $\mathcal{C} = \overline{\mathrm{span}}(\{\chi_{it},\ i \in \mathrm{N},\ t \in \mathrm{Z}\})$. Moreover, let us denote by $\mathcal{G}(\mathbf{x})$ the set of all the asymptotic aggregates of the x's. Proposition 2 implies that $\mathcal{G}(\mathbf{x}) \subseteq \mathcal{C}$, i.e. that if $\mathbf{a}_n(L)\mathbf{x}_{nt}$ converges, then the limit is an element of $\mathcal{C}$. In Proposition 3 we show that $\chi_{it} \in \mathcal{G}(\mathbf{x})$ for any $i \in \mathrm{N}$, so that $\mathcal{G}(\mathbf{x}) = \mathcal{C}$. Moreover, the proof provides a constructive procedure, based on the spectral density matrices $\boldsymbol{\Sigma}_n^x$, leading to a DAS $\{\underline{\mathbf{K}}_{ni}(L),\ n \in \mathrm{N}\}$ such that $\underline{\mathbf{K}}_{ni}(L)\mathbf{x}_{nt}$ converges to χ_{it}, for any $i \in \mathrm{N}$.

The construction will employ the dynamic principal components of the vector $\mathbf{x}_{nt}$. Let us recall that there exists an n-tuple of functions $\mathbf{p}_{nj}^x : [-\pi,\ \pi] \mapsto \mathrm{C}^n$, $j = 1, \ldots, n$, such that

(i) $\mathbf{p}_{j,n}^x(\theta)$ is a row eigenvector of $\boldsymbol{\Sigma}_n^x(\theta)$ corresponding to $\lambda_{nj}^x(\theta)$, i.e.,

$$\mathbf{p}_{nj}^x(\theta)\boldsymbol{\Sigma}_n^x(\theta) = \lambda_{nj}^x(\theta)\mathbf{p}_{nj}^x(\theta) \quad \text{for any} \quad \theta \in [-\pi,\ \pi];$$

(ii) $|\mathbf{p}_{nj}^x(\theta)|^2 = 1$ for any j and $\theta \in [-\pi,\ \pi]$;
(iii) $\mathbf{p}_{nj}^x(\theta)\tilde{\mathbf{p}}_{ns}^x(\theta) = 0$ for $j \neq s$ and any $\theta \in [-\pi,\ \pi]$;
(iv) $\mathbf{p}_{nj}^x$ is measurable on $[-\pi,\ \pi]$;
(see Brillinger, 1981, Chapter 9, and Forni and Lippi, 1998).

An n-tuple fulfilling properties (i) through (iv) will be called a set of *dynamic eigenvectors* of $\boldsymbol{\Sigma}_n^x$. A consequence of (ii) and (iv) is that the Fourier expansion

$$\mathbf{p}_{nj}^x(\theta) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left[\int_{-\pi}^{\pi}\mathbf{p}_{nj}(\theta)e^{-ik\theta}d\theta\right]e^{-ik\theta}$$

converges in the mean square. Defining

$$\underline{\mathbf{p}}^x_{nj}(L) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left[\int_{-\pi}^{\pi}\mathbf{p}_{nj}(\theta)e^{-ik\theta}d\theta\right]L^k,$$

the filter $\underline{\mathbf{p}}^x_{nj}(L)$ is square summable. For $j = 1,\ldots,n$, the scalar process $\{\underline{\mathbf{p}}^x_{nj}(L)\mathbf{x}_{nt},\ t \in \mathrm{Z}\}$, whose spectral density is

$$\mathbf{p}^x_{nj}\mathbf{\Sigma}^x_n\tilde{\mathbf{p}}^x_{nj} = \lambda^x_{nj},$$

will be called the j-th *dynamic principal component* of $\mathbf{x}_n$. A consequence of (iii) is that if $j \neq k$ then the j-th and the k-th principal components are orthogonal at any lead and lag.

In the sequel, given a function $\mathbf{f} : [-\pi,\ \pi] \mapsto \mathrm{C}^s$, whose components belong to $L_2([-\pi,\ \pi], \mathrm{C})$, we will denote by $\underline{\mathbf{f}}(L)$ the filter

$$\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left[\int_{-\pi}^{\pi}\mathbf{f}(\theta)e^{-ik\theta}d\theta\right]L^k$$

(a particular case is that of the eigenvectors $\mathbf{p}^x_{nj}$ and the filters $\underline{\mathbf{p}}^x_{nj}(L)$ defined above).

Now, let

$$x_{it} = \chi_{it,n} + \xi_{it,n},$$

where $\chi_{it,n}$ is the orthogonal projection of x_{it} onto the subspace of $L_2(\Omega, \mathcal{F}, P)$ spanned by the first q dynamic principal components of $\mathbf{x}_{nt}$, i.e.

$$\chi_{it,n} = \mathrm{Proj}\left(x_{it}\middle|\overline{\mathrm{span}}\left(\left\{\underline{\mathbf{p}}^x_{nj}(L)\mathbf{x}_{n\tau},\ j = 1,\ldots,q,\ \tau \in \mathrm{Z}\right\}\right)\right),$$

and $\xi_{it,n}$ the residual. To obtain an explicit expression for $\chi_{it,n}$, let us define $\mathbf{I}_n$ as the $n \times n$ identity matrix and observe that since

$$\mathbf{I}_n = \tilde{\mathbf{p}}^x_{n1}\mathbf{p}^x_{n1} + \tilde{\mathbf{p}}^x_{n2}\mathbf{p}^x_{n2} + \cdots + \tilde{\mathbf{p}}^x_{nn}\mathbf{p}^x_{nn}$$

(the vectors $\mathbf{p}^x_{nj}$ are an orthonormal system for $\mathbf{I}_n$),

$$\mathbf{x}_{nt} = \underline{\tilde{\mathbf{p}}}^x_{n1}(L)\underline{\mathbf{p}}^x_{n1}(L)\mathbf{x}_{nt} + \underline{\tilde{\mathbf{p}}}^x_{n2}(L)\underline{\mathbf{p}}^x_{n2}(L)\mathbf{x}_{nt} + \cdots + \underline{\tilde{\mathbf{p}}}^x_{nn}(L)\underline{\mathbf{p}}^x_{nn}(L)\mathbf{x}_{nt}.$$

Since different principal components are orthogonal at any lead and lag then

$$\chi_{it,n} = \left[\underline{\tilde{p}}^x_{n1,i}(L)\underline{\mathbf{p}}^x_{n1}(L) + \underline{\tilde{p}}^x_{n2,i}(L)\underline{\mathbf{p}}^x_{n2}(L) + \cdots + \underline{\tilde{p}}^x_{nq,i}(L)\underline{\mathbf{p}}^x_{nq}(L)\right]\mathbf{x}_{nt},$$

where $p^x_{nj,i}$ is the i-th component of $\mathbf{p}^x_{nj}$. Let us set

$$\mathbf{K}^x_{ni} = \tilde{p}^x_{n1,i}\mathbf{p}^x_{n1} + \tilde{p}^x_{n2,i}\mathbf{p}^x_{n2} + \cdots + \tilde{p}^x_{nq,i}\mathbf{p}^x_{nq}, \tag{4}$$

while $\underline{\mathbf{K}}^x_{ni}(L)$ is the corresponding filter.

PROPOSITION 3. *Suppose that Assumptions 1 through 4 hold. Then* $\{\underline{\mathbf{K}}^x_{ni}(L),\ n \in \mathrm{N}\}$ *is a DAS and*

$$\lim_{n\to\infty}\chi_{it,n} = \lim_{n\to\infty}\underline{\mathbf{K}}^x_{ni}(L)\mathbf{x}_{nt} = \chi_{it} \tag{5}$$

in mean square for all i and t.

PROOF. See the Appendix.

REMARK 1. Note that the filters $\underline{\mathbf{K}}^x_{ni}(L)$ result from a simple rule involving the dynamic eigenvectors of the matrices $\mathbf{\Sigma}^x_n$, with no intervention of the unobservable χ's and ξ's, and therefore can be estimated from the observed x's (see Section 7 below).

Now, let $\mathcal{U} = \overline{\mathrm{span}}(\{u_{jt},\ j = 1,\ \ldots,\ q,\ t \in \mathrm{Z}\})$. The inclusion $\mathcal{G}(\mathrm{x}) \subseteq \mathcal{U}$ immediately follows from Proposition 2. Proposition 4 proves the converse.

PROPOSITION 4. *Under Assumptions 1 through 4,*

$$\mathcal{G}(\mathrm{x}) = \mathcal{U}. \tag{6}$$

PROOF. See the Appendix.

The following is an immediate implication of Propositions 3 and 4.

COROLLARY. *Suppose that x_{it} can be represented as in (1), and that Assumptions 1 through 4 are fulfilled. Suppose that x_{it} admits the alternative representation*

$$x_{it} = b'_{i1}(L)u'_{1t} + b'_{i2}(L)u'_{2t} + \cdots + b'_{iq'}(L)u'_{q't} + \xi'_{it}, \tag{7}$$

and that Assumptions 1 through 4 are also fulfilled for (7). Then, $\chi'_{it} = x_{it} - \xi'_{it} = \chi_{it}$, so that $\xi'_{it} = \xi_{it}$. Moreover, $\mathcal{U} = \overline{\mathrm{span}}(\{u'_{jt},\ j = 1,\ldots,\ q';\ t \in \mathrm{Z}\})$, and therefore $q' = q$.

PROOF. Uniqueness of the common component follows from (5) and the fact that $\chi_{it,n}$ depends only on the x's. Equality (6) implies $\mathcal{U} = \overline{\mathrm{span}}(\{u'_{jt},\ j = 1,\ldots,\ q',\ t \in \mathrm{Z}\})$. QED

REMARK 2. An important consequence of the Corollary is that representation (1) is *non-redundant*, i.e. no other representation fulfilling Assumptions 1 through 4 is possible with a smaller number of factors. In the following example we have a common-idiosyncratic representation with one factor. However, since Assumption 4 is not fulfilled, a representation with zero factors fulfilling Assumptions 1 through 4 is possible. Consider the model

$$x_{it} = b_i u_t + \xi_{it},$$

where ξ_n is orthonormal white noise. Now suppose that the sequence of coefficients b_i, $i \in \mathrm{N}$, is square summable, i.e., that $\sum_{i=1}^{\infty} b_i^2 < \infty$. If $\{\mathbf{a}_n(L),\ n \in \mathrm{N}\}$ is a DAS, then

$$\mathbf{a}_n(L)\mathbf{x}_{nt} = \left(\sum_{i=1}^{n} a_{nj}(L) b_i\right) u_t + \sum_{i=1}^{n} a_{ni}(L)\xi_{it}.$$

The second term on the right hand side tends to zero by Proposition 2. The first term has variance

$$\int_{-\pi}^{\pi} |\sum_{i=1}^{n} a_{ni}(e^{-i\theta}) b_i|^2 d\theta \leq \left(\sum_{i=1}^{n} b_i^2\right)\left(\sum_{i=1}^{n}\int_{-\pi}^{\pi} |a_{ni}(e^{-i\theta})|^2 d\theta\right),$$

and therefore tends to zero by the assumption on the coefficients b_i and the definition of a DAS. Thus, neither u_t, nor the common component $\chi_{it} = b_i u_t$, can be recovered in this case. On the other hand, Assumption 4 does not hold since the first eigenvalue of $\mathbf{\Sigma}_n^X$ is $\sum_{i=1}^{n} b_i^2$, which is not divergent. Rather, the variables χ_{it} fulfill Assumption 3 and therefore, in spite of the non-zero correlation among different χ_{it}'s, they are idiosyncratic. In conclusion, a non-redundant common-idiosyncratic representation of x_{it}, i.e. a representation fulfilling Assumptions 1 through 4, must have the trivial common component $\chi_{it} = 0$ for all i.

5. Dynamic versus Static Analysis

The possibility of recovering the common components by aggregation has been studied in Chamberlain and Rothschild (1983) and Chamberlain (1983) for the model

$$x_i = c_{i1}v_1 + c_{i2}v_2 + \cdots + c_{iq}v_q + \rho_i, \tag{8}$$

which has no time dimension. Model (8) is "isomorphic" to model (1) under the assumptions that $b_{ij}(L)$ is constant and that ξ_n is a white noise process. If this is the case, the spectral density of $\mathbf{x}_n$, its eigenvalues and eigenvectors do not depend on θ, and coincide with the variance-covariance matrix of $\mathbf{x}_n$, its eigenvalues and eigenvectors, respectively (which are indeed the tools employed in Chamberlain and Rothschild's analysis). In this "static" case, our Assumptions 3 and 4 and Propositions 1 through 4 have a simpler form, in which reference to the frequency domain

can be avoided for eigenvalues and eigenvectors (as they are constant): for example, Proposition 1 in the static case simply states that the first q eigenvalues of the variance-covariance matrix diverge, and that the $(q+1)$-th one is bounded.

However, apart from the extreme white-noise case, there are specifications of (1) for which a "static" analysis may be tempting. Consider for instance

$$x_{it} = u_t + \alpha_i u_{t-1} + \xi_{it}, \tag{9}$$

with $\alpha_i = 1$ for i even, $\alpha_i = 0$ for i odd, and ξ_n orthonormal white noise. Defining $v_t = u_{t-1}$, model (9) might be thought of as isomorphic to model (8) with $q = 2$, with the consequence that only the variance-covariance matrix of $\mathbf{x}_n$ would be taken into consideration. The first two eigenvalues of this matrix diverge, while the third one is bounded (this results from direct analysis of the variance-covariance matrices), which is consistent with two static factors. However, this strategy is misleading, as shown by the fact that variance-covariance matrices do not reveal any distinction between (9) and

$$y_{it} = w_{1t} + \alpha_i w_{2t} + r_{it}, \tag{10}$$

where w_{1t} and w_{2t} are orthogonal at any lead and lag. By contrast, dynamic analysis, i.e. analysis of the eigenvalues of the spectral density matrices, yields:

(A) The process $\mathbf{y}_n$, generated by (10), has constant spectral density, so that dynamic and static analysis coincide. By Proposition 1, the first two eigenvalues diverge, whereas the third one is bounded, consistently with two factors.

(B) By Lemma 1, Appendix, the first eigenvalue of the spectral density matrix of $\mathbf{x}_n$ is not smaller than

$$\left|\sum_{i=1}^{n}(1 + \alpha_i e^{-i\theta})\right|^2 = |n(1 + \bar{\alpha}_n e^{-i\theta})|^2,$$

where $\bar{\alpha}_n = \sum_{i=1}^{n} \alpha_i / n$, and therefore diverges for any $\theta \in [-\pi,\ \pi]$, while the second dynamic eigenvalue is uniformly bounded; this is consistent with one dynamic factor. Thus, the difference between (9) and (10) is fully revealed.

Moreover, as soon as a model as simple as

$$x_{it} = \frac{1}{1 - \alpha_i L} u_t + \xi_{it}$$

is considered, with α_i drawn from the uniform distribution over $[0, 1]$, dynamic analysis reveals that the first eigenvalue diverges everywhere in $[-\pi,\ \pi]$, whereas the second one is uniformly bounded. By contrast, static analysis leads to the conclusion that all eigenvalues of the variance-covariance matrix diverge. This is consistent with an infinite number of static common factors, but also with completely different models, as for example

$$x_{it} = \rho_{it},$$

where the variables ρ_{it}, $i = 1, \ldots, n$ are mutually orthogonal (at any leads and lags) white noises such that $\text{var}(\rho_{it}) = i$.

6. The Choice of q

So far, we have assumed that q, the number of non-redundant common factors, is known. In practice of course, q is not predetermined, and also has to be selected from the data. Proposition 1 links the number of factors in (1) to the eigenvalues of the spectral density matrix of $\mathbf{x}_n$: precisely, if the number of factors is q and $\boldsymbol{\xi}$ is idiosyncratic, then the first q dynamic eigenvalues of Σ_n^x diverge a.e. in $[-\pi, \pi]$ whereas the $(q+1)$-th one is uniformly bounded. Forni and Lippi (1998) prove that the converse is also true: if the first q eigenvalues of Σ_n^x diverge a.e. in $[-\pi, \pi]$ and the $(q+1)$-th is uniformly bounded, then the x's admit a representation of the form (1) with q factors (and $\boldsymbol{\xi}$ idiosyncratic).

No formal testing procedure can be expected for selecting the number q of factors in finite sample situations. Even letting $T \to \infty$ does not help much. The definition of the idiosyncratic component indeed is of an asymptotic nature, where asymptotics are taken as $n \to \infty$, and there is no way a slowly diverging sequence (divergence, under the model, can be arbitrarily slow) can be told from an eventually bounded sequence (for which the bound can be arbitrarily large). Practitioners thus have to rely on a heuristic inspection of the eigenvalues against the number of series n.

More precisely, if T observations are available for a large number n of variables x_{it}, the spectral density matrices Σ_r^{xT}, $r \leq n$, can be estimated, and the resulting empirical dynamic eigenvalues λ_{rj}^{xT} computed for a grid of frequencies. The following two features of the eigenvalues computed from Σ_r^{xT}, $r = 1, \ldots, n$, should be considered as reasonable evidence that the data have been generated by (1), with q factors and that $\boldsymbol{\xi}$ is idiosyncratic:

(a) The average over θ of the first q empirical eigenvalues diverges, whereas the average of the $(q+1)$-th one is relatively stable.
(b) Taking $r = n$ there is a substantial gap between the variance explained by the q-th principal component and the variance explained by $(q+1)$-th one. A preassigned minimum, such as 10%, for the explained variance, could be used as a practical criterion for the determination of the number of common factors to retain. The 10% limit is used in the empirical exercise of Section 8.

To illustrate criteria (a) and (b), we have generated the following eight different factor models.

Static model, one factor:

$$x_{it} = a_i u_t + \sqrt{2}\xi_{it}. \qquad \text{M1a}$$

Static model, two factors:

$$x_{it} = a_i u_{1t} + b_i u_{2t} + \sqrt{2}\xi_{it}. \qquad \text{M1b}$$

Static model with delay, one factor:

$$\begin{aligned} x_{it} &= a_i u_{1t} + \sqrt{2}\xi_{it} && \text{for } i \text{ even} \\ x_{it} &= a_i u_{1t-1} + \sqrt{2}\xi_{it} && \text{for } i \text{ odd.} \end{aligned} \qquad \text{M2a}$$

Static with delay, two factors:

$$\begin{aligned} x_{it} &= a_i u_{1t} + b_i u_{2t} + \sqrt{2}\xi_{it} && \text{for } i \text{ even} \\ x_{it} &= a_i u_{1t-1} + b_i u_{2t-1} + \sqrt{2}\xi_{it} && \text{for } i \text{ odd.} \end{aligned} \qquad \text{M2b}$$

MA(1) model, one factor:

$$x_{it} = a_{0i} u_{1t} + a_{1i} u_{1t-1} + 2\xi_{it}. \qquad \text{M3a}$$

MA(1) model, two factors:

$$x_{it} = a_{0i} u_{1t} + a_{1i} u_{1t-1} + b_{0i} u_{2t} + b_{1i} u_{2t-1} + 2\xi_{it}. \qquad \text{M3b}$$

AR(1) common component, one factor:

$$x_{it} = \frac{a_i}{1 - c_i L} u_{1t} + \sqrt{2.5}\xi_{it}. \qquad \text{M4a}$$

AR(1) common component, two factors:

$$x_{it} = \frac{a_i}{1 - c_i L} u_{1t} + \frac{b_i}{1 - d_i L} u_{2t} + \sqrt{2.5}\xi_{it}. \qquad \text{M4b}$$

In all these models, u_{1t}, u_{2t}, a_i, a_{0i}, a_{1i}, b_i, b_{0i}, b_{1i} and ξ_{it} are i.i.d. standard normal deviates, while c_i and d_i are uniformly distributed over $[-0.8, 0.8]$, in order to ensure co-stationarity of the x's. Note that the idiosyncratic shocks are multiplied by a constant so that, on average, the cross sectional units have the same common-idiosyncratic variance ratio in all models. This ratio is 1/2 in the models with one factor and 1 in the models with two factors.

We have generated data from the above models with $n = 100$ and $T = 200$. Then, we have estimated the spectral density matrix for a grid of frequencies and computed the true spectral density matrix for the same frequencies.[5] Lastly, we have computed the eigenvalues of the upper-left $r \times r$ submatrix, $r = 1, \ldots, n$, both for theoretical and estimated spectral density matrices.

Figure 6.1 below reports the plot of the average over frequencies of the theoretical and estimated eigenvalues. On the horizontal axis we indicate the number of cross-sectional units r, which obviously is maximum when the whole sample $n = 100$ is considered. Features (a) and (b) emerge quite clearly for all models: the first q averaged eigenvalues exhibit an approximately constant positive slope, while the remaining ones are rather flat; moreover, the variance explained by the q-th principal component is substantially larger than the variance explained by the $(q+1)$-th, even for small r.

7. Estimation of the Common Component

7.1 Theory

Proposition 3 shows how the common component χ_{it} can be recovered, asymptotically, from the sequences $\underline{\mathbf{K}}_{ni}(L)\mathbf{x}_{nt}$, where the filters $\underline{\mathbf{K}}_{ni}(L)$ are obtained as functions of the eigenvectors $\mathbf{p}_{nj}(\theta)$, $\quad j = 1, \ldots, q$, associated with the spectral density matrices $\Sigma_n(\theta)$ (for notational simplicity, from now on, we will drop the superscript x when indicating spectral density, eigenvalues and eigenvectors associated with the x's). In practice, of course, these characteristics of the observed process are not available, and have to be replaced with empirical counterparts, based on finite realizations of the form

$$\mathbf{X}_n^T = (\mathbf{x}_{n1}, \quad \mathbf{x}_{n2}, \quad \ldots, \quad \mathbf{x}_{nT}).$$

Denote by $\Sigma_n^T(\theta)$ an estimator of the spectral density $\Sigma_n(\theta)$, based on the data in $\mathbf{X}_n^T$.

ASSUMPTION 5. Let $\sigma_{ij,n}(\theta)$ and $\sigma_{ij,n}^T(\theta)$ denote the i,j entries of $\Sigma_n(\theta)$ and $\Sigma_n^T(\theta)$ respectively. We suppose that $\sigma_{ij,n}^T(\theta)$ converges in probability to $\sigma_{ij,n}(\theta)$ uniformly in $[-\pi, \ \pi]$ for $T \to \infty$, i.e.

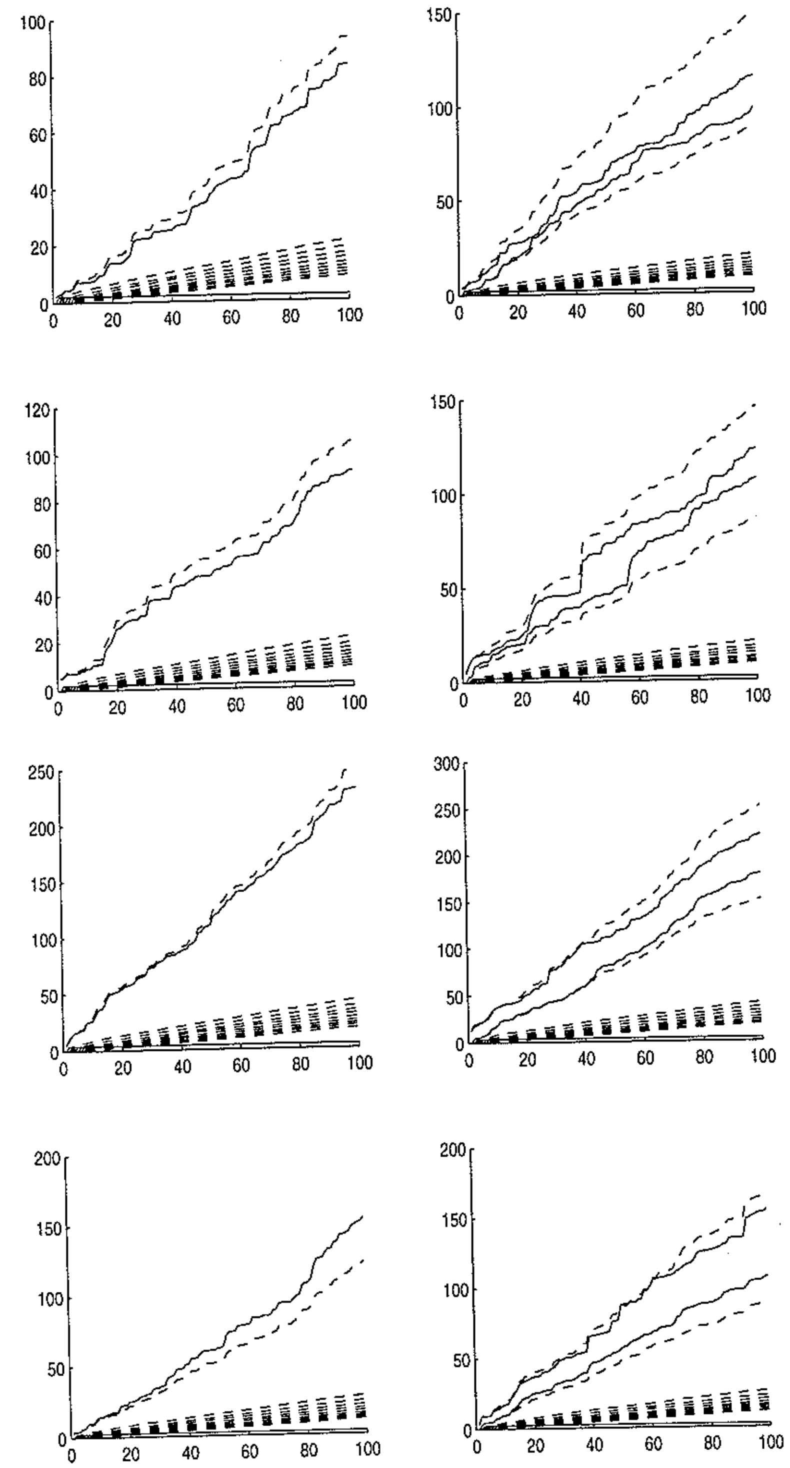
$$\lim_{T\to\infty} \mathrm{P}\{\max_{\theta\in[-\pi, \ \pi]} |\sigma_{ij,n}^T(\theta) - \sigma_{ij,n}(\theta)|\} = 0,$$

for any $i, j = 1, \ldots, n$.

Under Assumption 5, the estimated counterpart of $\mathbf{K}_{ni}(\theta)$ allows for a consistent reconstruction of the factor space. More precisely, we prove that the projection of

[5] The spectral density has been estimated using the method described in Section 7.2, Bartlett lag-window, size 14.

Figure 6.1. Dynamic eigenvalues averaged over frequencies, models (M1) through (M4)



Horizontal axis: r; vertical axis: variance$/2\pi$. Solid line: theoretical eigenvalues; dotted lines: estimated eigenvalues.

x_{it} onto the space spanned by the first q empirical principal components converges to the common component χ_{it}.

Assumption 5 is fulfilled under quite general conditions by lag-window or periodogram-smoothing spectral estimators (see, e.g., Grenander and Rosenblatt, 1957, p. 262).

Now, denote by $\lambda^T_{nj}(\theta)$ and $\mathbf{p}^T_{nj}(\theta)$, respectively, the eigenvalues and eigenvectors of the matrix $\mathbf{\Sigma}^T_n(\theta)$. Since eigenvalues and eigenvectors are continuous functions of the entries of the corresponding matrix, Assumption 5 implies that $\lambda^T_{nj}(\theta)$ and $\mathbf{p}^T_{nj}(\theta)$ converge to $\lambda_{nj}(\theta)$ and $\mathbf{p}_{nj}(\theta)$, respectively, in probability, uniformly in $\theta \in [-\pi,\ \pi]$, for $T \to \infty$.

Moreover, considering

$$\hat{\mathbf{K}}^T_{ni}(\theta) = \tilde{p}^T_{n1,i}(\theta)\mathbf{p}^T_{n1}(\theta) + \tilde{p}^T_{n2,i}(\theta)\mathbf{p}^T_{n2}(\theta) + \cdots + \tilde{p}^T_{nq,i}(\theta)\mathbf{p}^T_{nq}(\theta);$$

i.e., the empirical counterpart of $\mathbf{K}_{ni}(\theta)$, $i \leq q$, $\hat{\mathbf{K}}^T_{ni}(\theta)$ converges to $\mathbf{K}_{ni}(\theta)$ in probability, uniformly in $\theta \in [-\pi,\ \pi]$, for $T \to \infty$. Thus, for all $\epsilon > 0$ and $\eta > 0$, there exists $T_1 = T_1(n, \epsilon, \eta)$ such that, for all $T \geq T_1$,

$$\mathrm{P}[\sup_{\theta\in[-\pi,\ \pi]} \left| |\hat{\mathbf{K}}^T_{ni}(\theta) - \mathbf{K}_{ni}(\theta)| > \epsilon\right] \leq \eta.$$

Now, observe that, in principle, given the estimated spectral density matrix $\mathbf{\Sigma}^T_n(\theta)$, $\hat{\mathbf{K}}^T_{ni}(\theta)$ can be computed for any θ, so that each of the coefficients of the corresponding bilateral filter

$$\underline{\hat{\mathbf{K}}}^T_{ni}(L) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left[\int_{-\pi}^{\pi}\hat{\mathbf{K}}^T_{ni}(\theta)e^{-ik\theta}d\theta\right]L^k$$

can be obtained. However, in practice, the projection $\underline{\hat{\mathbf{K}}}^T_{ni}(L)\mathbf{x}_{nt}$ of x_{it} onto the space spanned by the first q empirical principal components cannot be computed, since, for $t \leq 0$ and $t > T$, $\mathbf{x}_{nt}$ is not available; $\underline{\hat{\mathbf{K}}}_{ni}(L)\mathbf{x}_{nt}$ actually has to be truncated at lag $t-1$ and lead $T-t$, respectively, yielding the finite-order filter $\underline{\hat{\mathbf{K}}}^{Tt}_{ni}(L)$. Due to this truncation, the common component χ_{it}, for fixed t, never can be recovered, even as n and T tend to infinity: indeed, part of its variance is lost because of the non-observability of $\mathbf{x}_{nt}$, $t \leq 0$ and $t > \tau$. We therefore restrict our attention to the "central part" of the observed series, i.e., to values of t of the form $t = t^*(T)$, with

$$0 < a \leq \liminf_{T\to\infty} \frac{t^*(T)}{T} \leq \limsup_{T\to\infty} \frac{t^*(T)}{T} \leq b < 1. \tag{11}$$

The following result then provides the empirical counterpart of Proposition 3.

PROPOSITION 5. *Assume that Assumptions 1 through 5 are satisfied. Then, for all $\epsilon > 0$ and $\eta > 0$, there exists $N_0(\epsilon, \eta)$ such that*

$$\mathrm{P}\left[|\underline{\hat{\mathbf{K}}}^{Tt}_{ni}(L)\mathbf{x}_{nt} - \chi_{it}| > \epsilon\right] \leq \eta$$

for all $t = t^*(T)$ satisfying (11), all $n \geq N_0$ and all T larger than some $T_0(n, \epsilon, \eta)$.

PROOF. For any $t \leq T$, we have

$$\begin{aligned}
&\mathrm{P}\left[|\underline{\hat{\mathbf{K}}}^{Tt}_{ni}(L)\mathbf{x}_{nt} - \chi_{it}| > \epsilon\right] \\
&\quad \leq \mathrm{P}\left[|(\underline{\hat{\mathbf{K}}}^{Tt}_{ni}(L) - \underline{\mathbf{K}}_{ni}(L))\mathbf{x}_{nt}| > \epsilon/2\right] + \mathrm{P}\left[|\underline{\mathbf{K}}_{ni}(L)\mathbf{x}_{nt} - \chi_{it}| > \epsilon/2\right] \\
&\quad = R^{Tt}_{n1} + R_{n2}, \quad \text{say.}
\end{aligned}$$

Proposition 3 ensures the existence of an $N_0(\epsilon, \eta)$ such that, for $n \geq N_0$, $R_{n2} =\leq \frac{\eta}{2}$. As for R^{Tt}_{n1}, it follows from Chebyshev's theorem that, for $T \geq T_1(n, \delta, \frac{\eta}{4})$,

$$\begin{aligned}
R^{Tt}_{n1} &\leq \mathrm{P}\left[|(\underline{\hat{\mathbf{K}}}^{Tt}_{ni}(L) - \underline{\mathbf{K}}_{ni}(L))\mathbf{x}_{nt}| > \epsilon/2 \text{ and } \sup_{\theta\in[-\pi,\ \pi]} |\hat{\mathbf{K}}^{Tt}_{ni}(\theta) - \mathbf{K}_{ni}(\theta)| \leq \delta\right] \\
&\quad + \mathrm{P}\left[\sup_{\theta\in[-\pi,\ \pi]} |\hat{\mathbf{K}}^{Tt}_{ni}(\theta) - \mathbf{K}_{ni}(\theta)| > \delta\right] \\
&\leq \frac{4}{\epsilon^2}\mathrm{E}\left[|(\underline{\hat{\mathbf{K}}}^{Tt}_{ni}(L) - \underline{\mathbf{K}}_{ni}(L))\mathbf{x}_{nt}|^2 \Big| \sup_{\theta\in[-\pi,\ \pi]} |(\hat{\mathbf{K}}^{Tt}_{ni}(\theta) - \mathbf{K}_{ni}(\theta))| \leq \delta\right] + \frac{\eta}{4}.
\end{aligned}$$

If the filter $\underline{\hat{\mathbf{K}}}^{Tt}_{ni}(L)$ and the observation $\mathbf{x}_{nt}$ were independent, then, in view of the classical properties of eigenvalues, the above expression would reduce to

$$\begin{aligned}
&\frac{4}{\epsilon^2}\mathrm{E}\left[|(\underline{\hat{\mathbf{K}}}^{Tt}_{ni}(L) - \underline{\mathbf{K}}_{ni}(L))\mathbf{x}_{nt}|^2 \Big| \sup_{\theta\in[-\pi,\ \pi]} |(\hat{\mathbf{K}}^{Tt}_{ni}(\theta) - \mathbf{K}_{ni}(\theta)) \leq \delta\right] + \frac{\eta}{4} \\
&\leq \frac{4}{2\pi\epsilon^2}\mathrm{E}\left[\int_{-\pi}^{\pi} (\underline{\hat{\mathbf{K}}}^{Tt}_{ni}(\theta) - \underline{\mathbf{K}}^{Tt}_{ni}(\theta))\mathbf{\Sigma}_n(\theta)(\underline{\hat{\tilde{\mathbf{K}}}}^{Tt}_{ni}(\theta) - \underline{\tilde{\mathbf{K}}}^{Tt}_{ni}(\theta))d\theta\right. \\
&\qquad \left. \Big| \sup_{\theta\in[-\pi,\ \pi]} |(\hat{\mathbf{K}}^{Tt}_{ni}(\theta) - \mathbf{K}_{ni}(\theta))| \leq \delta\right] + \frac{\eta}{4} \\
&\leq \frac{2\delta^2}{\epsilon^2}\int_{-\pi}^{\pi} \lambda_{n1}(\theta)d\theta + \frac{\eta}{2}.
\end{aligned}$$

Thus, for $n \geq N_0(\epsilon, \eta)$, and $T \geq T_1(n, \delta, \eta/4)$, with $\frac{\delta^2=\epsilon^2\eta}{8\int_{-\pi}^{\pi}\lambda_{n1}(\theta)d\theta}$, we would obtain $R_{n1}^{Tt} + R_{n2}^{Tt} \leq \eta$ for all $t = t^*(T)$ satisfying (11). The proposition follows.

This reasoning is essentially correct, and it carries the basic idea of the proof below, since the dependence between $\underline{\hat{\mathbf{K}}}_{ni}^{Tt}(L)$ and $\mathbf{x}_{nt}$ vanishes as $T \to \infty$. A formal treatment, however, requires a slightly more elaborate argument: see the Appendix.

7.2 The proposed estimator

In practice, the estimator of the common components can be constructed by running n OLS regressions (equation by equation) of the observations on present, past and future of the estimated dynamic principal components $\underline{\mathbf{p}}_{nj}^{T}(L)\mathbf{x}_{nt}$, $j = 1, \ldots, q$. Proposition 5 applies provided that the estimated eigenvectors fulfill Assumption 5.

We proceed as follows. For a fixed integer M, we compute the sample covariance matrix $\mathbf{\Gamma}_{nk}^{T}$ of $\mathbf{x}_{nt}$ and $\mathbf{x}_{nt-k}$ for $k = 0, \ldots, M$ and the $2M+1$ points discrete Fourier transform of the truncated bilateral sequence $\mathbf{\Gamma}_{n,-M}^{T}, \ldots, \mathbf{\Gamma}_{n0}^{T}, \ldots, \mathbf{\Gamma}_{nM}^{T}$, where $\mathbf{\Gamma}_{n,-k} = \mathbf{\Gamma}_{nk}'$. This implies computing

$$\mathbf{\Sigma}_n^T(\theta_h) = \sum_{k=-M}^{M} \mathbf{\Gamma}_{nk}^{T} \omega_k e^{-ik\theta_h}, \; \theta_h = 2\pi h/(2M+1), \; h = 0, \ldots, 2M,$$

where $\omega_k = 1 - \frac{k}{(M+1)}$ are the weights corresponding to the Bartlett lag window of size M. Provided that $M \to \infty$ and $M/T \to 0$ as $T \to \infty$, $\mathbf{\Sigma}_n^T(\theta_h)$ satisfies Assumption 5.

Then we compute the first q eigenvectors $\mathbf{p}_{nj}^{T}$, $j = 1, \ldots, q$, of $\mathbf{\Sigma}_n^T(\theta_h)$, for $h = 0, \ldots, 2M$. The proposed estimator of the filter $\underline{\mathbf{p}}_{nj}(L)$, $j = 1, \ldots, q$, is constructed from the inverse discrete Fourier transform of the vector

$$\left(\mathbf{p}_{nj}^{T}(\theta_1), \quad \cdots, \quad \mathbf{p}_{nj}^{T}(\theta_{2M})\right),$$

i.e., from the computation of

$$\underline{\mathbf{p}}_{nj,k}^{T} = \frac{1}{2M+1}\sum_{h=0}^{2M} \mathbf{p}_{nj}^{T}(\theta_h) e^{ik\theta_h}$$

for $k = -M, \ldots, M$. The estimator of the filter is given by

$$\underline{\mathbf{p}}_{nj}^{T}(L) = \sum_{k=-M}^{M} \underline{\mathbf{p}}_{nj,k}^{T} L^k. \tag{12}$$

It must be pointed out that an estimator of the common components could be obtained by computing $\underline{\hat{\mathbf{K}}}_{ni}^{Tt}(L)$, as suggested by (4), then applying this filter to $\mathbf{x}_{nt}$. This estimator however gives poor results in small samples. The alternative procedure we suggest here, i.e., the OLS estimates of the observations regressed on present, past and future of the estimated dynamic principal components, seems preferable in practice. Note that, for $M = 0$, $\mathbf{p}_{nj}^{T}(L)$ is simply the j-th eigenvector of the (estimated) variance-covariance matrix of $\mathbf{x}_{nt}$: the dynamic principal components then reduce to the static principal components.

In order to render our procedure operational, we need a rule for fixing M as well as the number of leads (s) and lags (g) of the principal components to be retained in the regressions. We propose the following rule. First, take a maximum value for M, s and g, say $M_0(T)$, such that $M_0(T) \to \infty$ and $M_0(T)/T \to 0$ as $T \to \infty$. Second, estimate all of the specifications with $0 \leq M \leq M_0(T)$, $0 \leq s \leq M_0(T)$, $0 \leq g \leq M_0(T)$, and choose the one minimizing some dynamic specification criterion. Here we propose

$$\frac{T}{n}\sum_{i=1}^{n} \log \hat{\sigma}_i + 2q(s+g+1), \tag{13}$$

where $\hat{\sigma}_i$ is the estimated variance of the residuals of equation i. This criterion is the cross-sectional average of AICs. Note that we cannot use the multivariate AIC, since the determinant of the covariance matrix of the residuals is very poorly estimated for large n (the estimate is 0 for $n > T$). Neither do we propose BIC, since we found that a richer dynamic specification such as the one implied by the AIC criterion gives better results for our simulated models.

7.3 Simulation results

In order to evaluate the performance of our estimation procedure for finite values of n and T, we have carried out Monte Carlo experiments on models (M1b), (M2b), (M3b) and (M4b) of Section 6. We generated data from each model with $n = 10, 20, 50, 100$ and $T = 20, 50, 100, 200$ and applied the estimation procedure described in Section 7.2 with $M_0(T) = \text{round}[\frac{\sqrt{T}}{4}]$. Each experiment was replicated 400 times.

We measured the performance of our estimator, $\hat{\chi}_{it}$, by means of the criterion

$$R(\hat{\chi}, \chi) = \frac{\sum_{i,t}(\hat{\chi}_{it} - \chi_{it})^2}{\sum_{i,t}\chi_{it}^2}.$$

Table 7.1 reports the average and the standard deviation (in brackets) of this statistic across the experiments.

For all models, we see that the fit improves as both n and T increase. To better appreciate the results, we added a line reporting $R(\bar{\chi}, \chi)$, where $\bar{\chi}_{it}$ is the unfeasible estimate of the common components obtained by using the unobservable true common factors u_{jt} in place of the dynamic principal components as the regressors; $\bar{\chi}_{it}$ is computed only for $n = 100$. The average AIC criterion (13) is used for dynamic specification. Note that for the autoregressive model (M4b), the results obtained with $n \geq 50$ are comparable with those obtained with the true factors.

Table 7.1. Average and standard deviation (in brackets) of $R(\hat{\chi}, \chi)$ across 400 experiments

	$T = 20$	$T = 50$	$T = 100$	$T = 200$
Model (M1b)				
$n = 10$	0.636 (0.380)	0.344 (0.173)	0.273 (0.125)	0.249 (0.095)
$n = 20$	0.462 (0.273)	0.188 (0.070)	0.145 (0.042)	0.124 (0.033)
$n = 50$	0.363 (0.216)	0.106 (0.027)	0.073 (0.014)	0.057 (0.010)
$n = 100$	0.320 (0.214)	0.084 (0.018)	0.052 (0.008)	0.036 (0.005)
$R(\bar{\chi}, \chi)$ with $n = 100$	0.197 (0.105)	0.062 (0.012)	0.031 (0.005)	0.015 (0.002)
Model (M2b)				
$n = 10$	0.754 (0.466)	0.502 (0.206)	0.406 (0.169)	0.329 (0.130)
$n = 20$	0.651 (0.273)	0.383 (0.141)	0.259 (0.081)	0.198 (0.063)
$n = 50$	0.569 (0.190)	0.291 (0.085)	0.177 (0.049)	0.122 (0.032)
$n = 100$	0.563 (0.167)	0.269 (0.071)	0.151 (0.041)	0.096 (0.027)
$R(\bar{\chi}, \chi)$ with $n = 100$	0.347 (0.113)	0.103 (0.019)	0.052 (0.008)	0.025 (0.004)
Model (M3b)				
$n = 10$	0.718 (0.263)	0.500 (0.151)	0.384 (0.110)	0.329 (0.087)
$n = 20$	0.637 (0.225)	0.389 (0.115)	0.277 (0.064)	0.208 (0.043)
$n = 50$	0.568 (0.173)	0.306 (0.077)	0.196 (0.046)	0.136 (0.032)
$n = 100$	0.556 (0.154)	0.283 (0.071)	0.170 (0.043)	0.113 (0.028)
$R(\bar{c}, c)$ with $n = 100$	0.362 (0.112)	0.106 (0.019)	0.052 (0.007)	0.026 (0.003)
Model (M4b)				
$n = 10$	0.684 (0.393)	0.440 (0.214)	0.334 (0.137)	0.281 (0.106)
$n = 20$	0.549 (0.222)	0.289 (0.095)	0.208 (0.057)	0.169 (0.044)
$n = 50$	0.485 (0.193)	0.196 (0.053)	0.132 (0.025)	0.097 (0.016)
$n = 100$	0.456 (0.156)	0.165 (0.029)	0.107 (0.014)	0.072 (0.009)
$R(\bar{\chi}, \chi)$ with $n = 100$	0.405 (0.118)	0.183 (0.028)	0.112 (0.015)	0.066 (0.007)

Finally, in order to evaluate the ability of the average AIC criterion (13) in choosing the best dynamic specification, we computed R^*, i.e., the minimum of R over k, g and s, for each of the experiments of Table 7.1. The average R^* is reported in Table 7.2. The results are very good: comparing Table 7.2 with Table 7.1, we see that, for $T \geq 50$ and $n \geq 20$, R is very close to R^* for all models.

Table 7.2. Average R^*

	$T = 20$	$T = 50$	$T = 100$	$T = 200$
Model (M1b)				
$n = 10$	0.477	0.317	0.263	0.247
$n = 20$	0.301	0.182	0.145	0.123
$n = 50$	0.212	0.106	0.073	0.057
$n = 100$	0.184	0.083	0.052	0.036
Model (M2b)				
$n = 10$	0.611	0.440	0.367	0.304
$n = 20$	0.514	0.335	0.242	0.191
$n = 50$	0.436	0.264	0.169	0.120
$n = 100$	0.416	0.243	0.148	0.094
Model (M3b)				
$n = 10$	0.595	0.442	0.356	0.313
$n = 20$	0.506	0.340	0.259	0.191
$n = 50$	0.432	0.271	0.188	0.133
$n = 100$	0.416	0.253	0.163	0.111
Model (M4b)				
$n = 10$	0.527	0.385	0.312	0.269
$n = 20$	0.402	0.262	0.199	0.165
$n = 50$	0.333	0.182	0.128	0.096
$n = 100$	0.305	0.158	0.106	0.072

8. Empirical Illustration

In this Section, we illustrate our method by estimating the generalyzed dynamic factor model for a panel of annual real output growth of 49 US states from 1948 through 1993. The objective is to estimate the "national component" of the US business cycle, and to extract information on its dynamic structure.

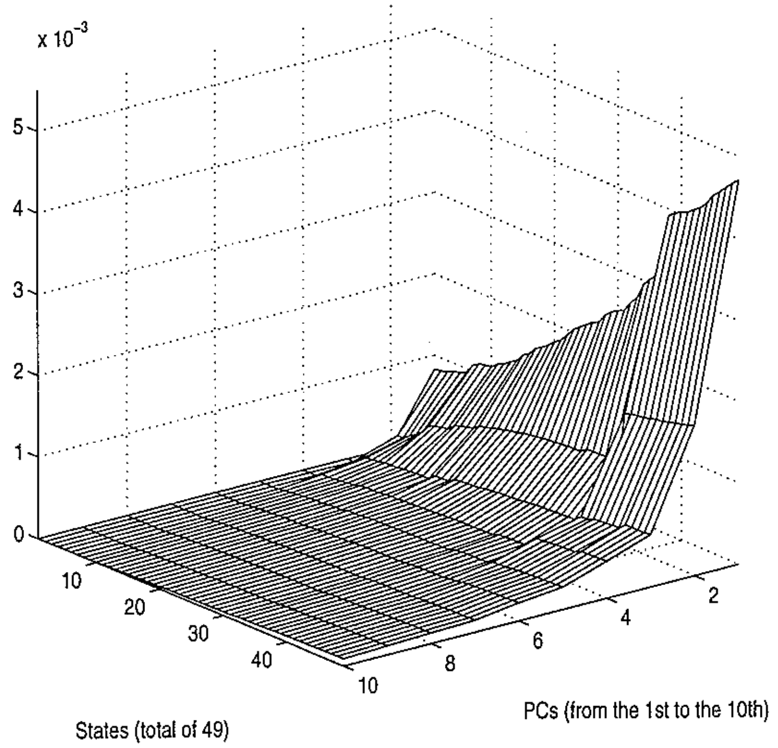
We proceed in two steps.

Step 1. We first identify q using the procedure outlined in Section 6. Figure 8.1 is a three-dimensional plot of the first ten averaged (over frequencies) eigenvalues for cross-sectional groups of different size (random order).

Note that two eigenvalues are increasing with n, while the remaining ones have a flat shape. When the whole cross-section is considered, the first principal component explains on average 52% of the variance, the second one 20%, and the third one, 7%. Thus, according to both informal criteria suggested in Section 6, we may conclude for a two common-factor model.

Step 2. Having identified $q = 2$, we then proceed to the estimation of the common component following the methodology outlined in Section 7.2. The lag selection criterion suggests $s = 1$, $g = 2$. Regressions on leads and lags of the first two principal components produce an average R^2 of 59.3%, with a standard deviation of 15.7%. Figure 8.2 reports the estimates for six large states. Due to the heterogenous

Figure 8.1. Ten largest dynamic eigenvalues (increasing n-averaged over frequencies)



propagation mechanism, the common components have different variances and different turning points. However, all states have downturns corresponding to the two major oil shocks. A detailed analysis of turning points is beyond the scope of this paper, but the empirical illustration conducted here indicates potentially interesting applications of our method for the analysis of regional business cycles and their synchronization.

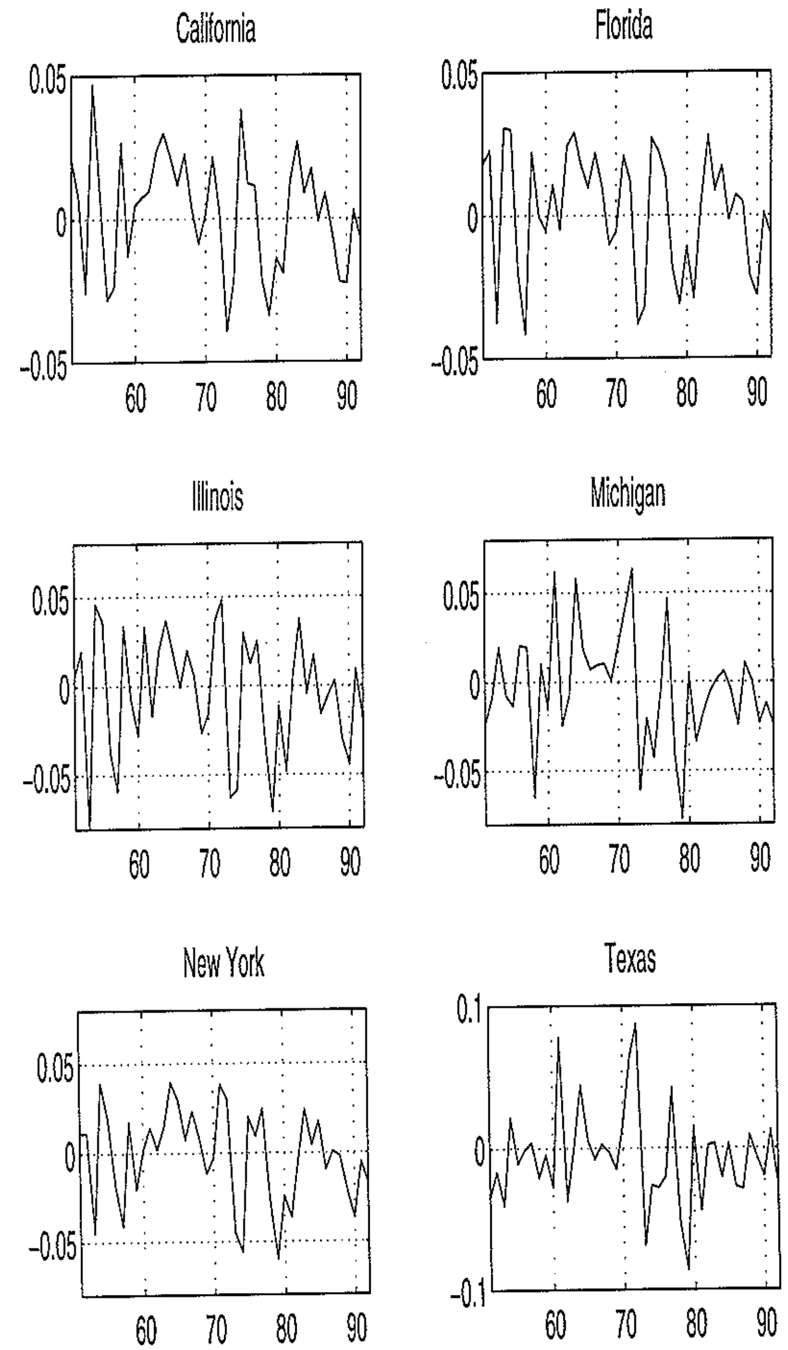
From the estimated common components we also computed the average spectral densities of the national and state-specific component of output changes. These are presented in Figure 8.3.

From this last exercise, we can observe that output fluctuations in US states have a large common component with a clear peak at cycles of period ranging from 6 to 9 years. The idiosyncratic component is not only small, but also has no cyclical shape.

9. Summary and Discussion

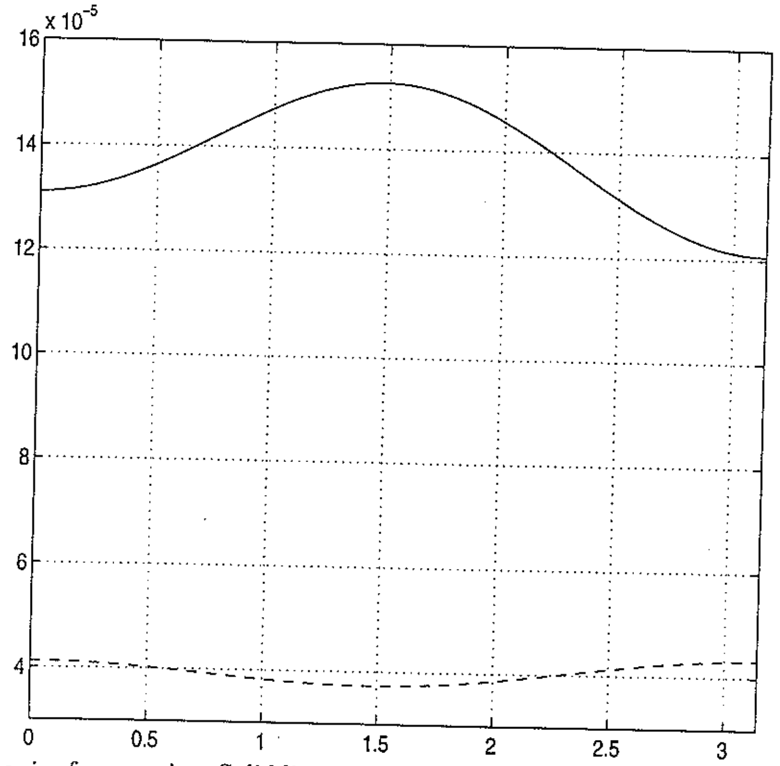
The generalized dynamic factor model analyzed in this paper is novel to the liter-

Figure 8.2 Common component of six representative US states 1951-1992



Horizontal axis: years.

Figure 8.3 Average spectra of the common and idiosyncratic components



Horizontal axis: frequencies. Solid line: national component; dotted lines: idiosyncratic component.

ature since it allows for an infinite moving average representation of the common component and for non-orthogonal idiosyncratic components. We have shown that, although for a finite cross-sectional dimension this model is not identified in general, asymptotic identification conditions can be established as the cross-sectional dimension goes to infinity. These identification conditions are given on the spectral density matrix of the process and therefore allow us to distinguish between static and lagged factors.

For large cross-sections, dynamic factor models cannot be estimated on the basis of traditional likelihood based methods. We have proposed a method of estimation which is appropriate in such situations and simple to implement. This method allows for consistent estimates of the components as the cross section and the time dimension go to infinity at some rate. The common components are computed as the projections of the observations onto the leads and lags of the dynamic principal components of the observations and the idiosyncratic components are derived as the orthogonal residuals.

Consistent estimation of the components is essential for "historical analysis" and for the identification of the factors driving common and idiosyncratic dynamics. In the empirical illustration we show, for example, the estimates of the "national

component" of output growth of US states. This analysis can be exploited to study turning points in regional business cycles and it is a first necessary step for identifying the factors, i.e., the shocks driving the components of the business cycle (the factors are only identified up to an orthogonal rotation). This analysis is beyond the scope of this paper, but, given our results on estimation of the components, it can be easily conducted along the lines suggested by Forni and Reichlin (1998).

Notice that the model and the estimation strategy proposed here cannot be used for forecasting purposes without imposing some further restrictive assumptions on the dynamics of the factors and modifying the bilateral filters used for estimation. Forecasting is the objective of a recent related paper by Stock and Watson (1998).

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APPENDIX

LEMMA 1. *The following inequalities hold for any* $\theta \in [-\pi, \pi]$, j *and* n:
$\lambda^x_{nj}(\theta) \leq \lambda^\chi_{nj}(\theta) + \lambda^\xi_{nj}(\theta)$;
$\lambda^x_{nj}(\theta) \geq \lambda^\chi_{nj}(\theta)$;
$\lambda^x_{nj}(\theta) \geq \lambda^\xi_{nj}(\theta)$.

PROOF. Recall that the eigenvalues λ_k, $k = i, \dots, n$ of a complex non-negative definite $n \times n$ hermitian matrix $\mathbf{\Psi}$ solve the following minimax problem:

$$\lambda_k = \min_{\mathbf{c}_1,\dots,\mathbf{c}_{k-1}} \max\{\mathbf{b}\mathbf{\Psi}\tilde{\mathbf{b}} \big| |\mathbf{b}| = 1, \mathbf{b} \perp \mathbf{c}_1, \dots, \mathbf{b} \perp \mathbf{c}_{k-1}\}$$

where $\mathbf{c}_1, \dots, \mathbf{c}_{k-1}$ and $\mathbf{b}$ belong to C^n (see Brillinger, 1981, p. 84, Exercise 3.10.16). Then the lemma follows immediately from the fact that $\mathbf{\Sigma}^x_n = \mathbf{\Sigma}^\chi_n + \mathbf{\Sigma}^\xi_n$. QED

PROOF OF PROPOSITION 1. The statement on the first q eigenvalues of $\mathbf{\Sigma}^x_n$ follows from the second inequality of Lemma 1. The statement on the $(q+1)$-th one follows from the first inequality and the fact the $(q+1)$-th eigenvalue of $\mathbf{\Sigma}^\chi_n$ vanishes at any frequency. QED

To prove Propositions 3 and 4 we need some intermediate lemmas.

LEMMA 2. *Let* $p^x_{nj,i}$ *be the i-th coordinate of* $\mathbf{p}^x_{nj}$. *For* $j \leq q$, $\lim_n p^x_{nj,i}(\theta) = 0$ *for* θ *a.e. in* $[-\pi, \pi]$.
PROOF. Let $\mathbf{P}^x_n$ be the $n \times n$ matrix having the eigenvectors $\mathbf{p}^x_{nj}$, $j = 1, \dots, n$ on the rows. From the identity $\tilde{\mathbf{P}}^x_n \mathrm{diag}\,(\lambda^x_{n1}, \ \lambda^x_{n2}, \ \cdots, \ \lambda^x_{nn})\,\mathbf{P}^x_n = \mathbf{\Sigma}^x_n$, we obtain

$$\sum_{j=1}^{q} |p^x_{nj,i}(\theta)|^2 \lambda^x_{nj}(\theta) + \sum_{j=q+1}^{n} |p^x_{nj,i}(\theta)|^2 \lambda^x_{nj}(\theta) = \sigma^x_i(\theta),$$

where σ^x_i is the spectral density of x_{it}. By Proposition 1, $\lambda^x_{nj}(\theta)$ diverges, θ a.e. in $[-\pi, \pi]$, for $j \leq q$, so that, for $j \leq q$ and θ a.e. in $[-\pi, \pi]$, $|p^x_{nj,i}(\theta)|$ converges to zero. QED

LEMMA 3. *Suppose that* $\{(v_{n1t} \ \ v_{n2t} \ \ \cdots \ \ v_{nqt})', \ t \in Z\}$ *is a q-dimensional orthonormal white noise for any* $n \in N$, *and that, given the orthogonal projection,*

$$(v_{n1t}, \ v_{n2t}, \ \cdots, \ v_{nqt})' = \mathbf{A}_n(L)\,(u_{1t}, \ u_{2t}, \ \cdots, \ u_{qt})' + \mathbf{R}_{nt}, \qquad (14)$$

with $\mathbf{R}_{nt}$ *orthogonal to* $(u_{1t-k}, \ u_{2t-k}, \ \cdots, \ u_{qt-k})'$ *for any* $k \in Z$, *the* $q \times q$ *spectral density matrix of* $\mathbf{R}_{nt}$ *converges to zero a.e. in* $[-\pi, \pi]$. *Consider the orthogonal projection*

$$(u_{1t}, \ u_{2t}, \ \cdots, \ u_{qt})' = \tilde{\mathbf{A}}_n(L^{-1})\,(v_{n1t}, \ v_{n2t}, \ \cdots, \ v_{nqt})' + \mathbf{S}_{nt}, \qquad (15)$$

with $\mathbf{S}_{nt}$ *orthogonal to* $(v_{n1t-k}, \ v_{n2t-k}, \ \cdots, \ v_{nqt-k})'$ *for any* $k \in Z$. *Then, the spectral density of* $\mathbf{S}_{nt}$ *converges to zero a.e. in* $[-\pi, \pi]$.
PROOF. Since $\mathbf{A}_n(e^{-i\theta})\tilde{\mathbf{A}}_n(e^{i\theta})$ is Hermitian non-negative definite, there exists a unitary matrix $\mathbf{M}_n(e^{-i\theta})$ be unitary and such that

$$\mathbf{M}_n(e^{-i\theta})\mathbf{A}_n(e^{-i\theta})\tilde{\mathbf{A}}_n(e^{i\theta})\tilde{\mathbf{M}}_n(e^{i\theta}) = \mathrm{diag}\,(\nu_{n1}(\theta), \ \nu_{n2}(\theta), \ \cdots, \nu_{nq}(\theta)),$$

the functions ν_{nj} being the non-negative eigenvalues of $\mathbf{A}_n(e^{-i\theta})\tilde{\mathbf{A}}_n(e^{i\theta})$. From (14),

$$\begin{aligned}&\mathbf{M}_n(L)\,(v_{n1t}, \ v_{n2t}, \ \cdots, \ v_{nqt})' \\ &\quad = \mathbf{M}_n(L)\mathbf{A}_n(L)\,(u_{1t}, \ u_{2t}, \ \cdots, \ u_{qt})' + \mathbf{M}_n(L)\mathbf{R}_{nt}.\end{aligned} \qquad (16)$$

Since the entries of $\mathbf{M}_n(e^{-i\theta})$ are bounded uniformly in n, the spectral density of $\mathbf{M}_n(L)\mathbf{R}_{nt}$ tends to zero a.e. in $[-\pi, \pi]$. Thus, taking the spectral densities of both sides in (16),

$$\lim_{n\to\infty} (1 - \nu_{nj}(\theta)) = 0$$

a.e. in $[-\pi, \pi]$ for $j = 1, \dots, q$. Uniform boundedness of the entries of $\mathbf{M}_n(e^{-i\theta})$ implies that

$$\lim_{n\to\infty} \left[\mathbf{I}_q - \mathbf{A}_n(e^{-i\theta})\tilde{\mathbf{A}}_n(e^{i\theta})\right] = 0,$$

a.e. in $[-\pi, \pi]$. Moreover, since the eigenvalues of $\mathbf{A}_n(e^{-i\theta})\tilde{\mathbf{A}}_n(e^{i\theta})$ coincide with the eigenvalues of $\tilde{\mathbf{A}}_n(e^{i\theta})\mathbf{A}_n(e^{-i\theta})$ (see Brillinger, 1881, Theorem 3.7.2, p. 72),

$$\lim_{n\to\infty} \left[\mathbf{I}_q - \tilde{\mathbf{A}}_n(e^{i\theta})\mathbf{A}_n(e^{-i\theta})\right] = 0$$

a.e. in $[-\pi, \pi]$, and therefore

$$\lim_{n\to\infty} \left[\mathbf{I}_q - \tilde{\mathbf{A}}_n(e^{i\theta})\mathbf{A}_n(e^{-i\theta})\right]^2 = 0 \qquad (17)$$

a.e. in $[-\pi, \pi]$. From (15), using (14),

$$\left[\mathbf{I}_q - \tilde{\mathbf{A}}_n(L^{-1})\mathbf{A}_n(L)\right](u_{1t}, \ u_{2t}, \ \cdots, \ u_{qt})' - \tilde{\mathbf{A}}_n(L^{-1})\mathbf{R}_{nt} = \mathbf{S}_{nt}.$$

The spectral density of the first term on the left hand side tends to zero a.e. in $[-\pi, \pi]$ by (17). Uniform boundedness of the entries of $\mathbf{A}_n(e^{-i\theta})$ implies that the spectral density of $\tilde{\mathbf{A}}_n(L^{-1})\mathbf{R}_{nt}$ tends to zero a.e. in $[-\pi, \pi]$. Thus, the spectral density of $\mathbf{S}_{nt}$ tends to zero a.e. in $[-\pi, \pi]$ as well. QED

With no loss of generality we can assume that

ASSUMPTION A. $\lambda^x_{nj}(\theta) \geq 1$ for any j, n and $\theta \in [-\pi,\ \pi]$.

For, possibly by embedding $L_2(\Omega, \mathcal{F}, P)$ into a larger space, we can assume that $L_2(\Omega, \mathcal{F}, P)$ contains a double sequence $\{\phi_{it},\ i \in \mathrm{N},\ t \in \mathrm{Z}\}$ such that $\phi_{it} \perp \overline{\mathrm{span}}(\{\xi_{it},\ i \in \mathrm{N},\ t \in \mathrm{Z}\})$, $\phi_{it} \perp \mathcal{U}$ for any i, $\mathrm{var}(\phi_{it}) = 1$ for any i, and $\phi_{it} \perp \phi_{jt-k}$ for any t, k and $i \neq j$. Defining $\hat{\xi}_{it} = \xi_{it} + \phi_{it}$, and

$$y_{it} = \chi_{it} + \hat{\xi}_{it}, \tag{18}$$

for $i \in \mathrm{N}$ and $t \in \mathrm{Z}$, we obviously have:

(1) Model (18) fulfills Assumptions 1 through 4, with $\mathbf{\Sigma}^{\hat{\xi}}_n(\theta) = \mathbf{\Sigma}^{\xi}_n(\theta) + \mathbf{I}_n$, $\mathbf{\Sigma}^y_n(\theta) = \mathbf{\Sigma}^x_n(\theta) + \mathbf{I}_n$, and therefore $\lambda^{\hat{\xi}}_{nj}(\theta) = \lambda^{\xi}_{nj}(\theta) + 1$, $\lambda^y_{nj}(\theta) = \lambda^x_{nj}(\theta) + 1$. Moreover, $\mathbf{p}^y_{nj} = \mathbf{p}^x_{nj}$ for any n and j, so that $\mathbf{K}^y_{ni} = \mathbf{K}^x_{ni}$ for any n and i.
(2) As a consequence, if we prove that $\{\underline{\mathbf{K}}^y_{ni}(L),\ n \in \mathrm{N}\}$ is a DAS and that $\lim_{n\to\infty} \underline{\mathbf{K}}^y_{ni}(L)\mathbf{y}_{nt} = \chi_{it}$, then, since $\lim_n \underline{\mathbf{K}}^y_{ni}(L)\phi_{nt} = 0$, we have $\lim_{n\to\infty} \underline{\mathbf{K}}^x_{ni}(L)\mathbf{x}_{nt} = \chi_{nt}$.

Under Assumption A the function $\mu^x_{nj}(\theta) = [\lambda^x_{nj}(\theta)]^{-1/2}$ is defined for any $\theta \in [-\pi,\ \pi]$, is bounded and therefore has a mean-square convergent Fourier representation. Let us denote by $\underline{\mu}^x_{jn}(L)$ the corresponding square-summable filter.

LEMMA 4. *Under Assumptions 1 through 4 and Assumption A, setting* $v_{njt} = \underline{\mu}^x_{nj}(L)\underline{\mathbf{p}}^x_{nj}(L)\mathbf{x}_{nt}$, *for* $j = 1,\ \ldots,\ q$, *the vector* $(v_{n1t},\ v_{n2t},\ \cdots,\ v_{nqt})'$ *is orthonormal white noise. Moreover, given the orthogonal decomposition*

$$(v_{n1t},\ v_{n2t},\ \cdots,\ v_{nqt})' = \mathbf{A}_n(L)(u_{1t},\ u_{2t},\ \cdots,\ u_{qt})' + \mathbf{R}_{nt}, \tag{19}$$

the spectral density of $\mathbf{R}_{nt}$ *converges to zero a.e. in* $[-\pi,\ \pi]$.
PROOF. The first statement follows from the definition of dynamic eigenvectors. Then note that, since the χ's belong to $\mathcal{U}$ and the ξ's are orthogonal to $\mathcal{U}$,

$$v_{njt} = \underline{\mu}^x_{nj}(L)\underline{\mathbf{p}}^x_{nj}(L)\mathbf{x}_{nt} = \underline{\mu}^x_{nj}(L)\underline{\mathbf{p}}^x_{nj}(L)\chi_{nt} + \underline{\mu}^x_{nj}(L)\underline{\mathbf{p}}^x_{nj}(L)\boldsymbol{\xi}_{nt}$$

is the j-th orthogonal projection in (19). The spectral density of the second term on the right hand side is

$$[\mu^x_{nj}(\theta)]^2\mathbf{p}^x_{nj}(\theta)\Sigma^{\xi}_n(\theta)\tilde{\mathbf{p}}^x_{nj}(\theta) \leq [\mu^x_{nj}(\theta)]^2\lambda^{\xi}_n(\theta),$$

which converges to zero a.e. in $[-\pi,\ \pi]$ by Assumptions 3 and 4. QED

Incidentally, note that Assumption 4, together with $\mu^x_{nj}(\theta) \leq 1$, i.e., Assumption A, implies, by the Lebesgue dominated convergence theorem, that $\underline{\mu}^x_{nj}(L)\underline{\mathbf{p}}^x_{nj}(L)$ is a DAS.

PROOF OF PROPOSITIONS 3 AND 4. We have

$$x_{it} = \chi_{it} + \xi_{it} = \chi_{it,n} + \xi_{it,n}, \tag{20}$$

and

$$\chi_{it,n} = \underline{\mathbf{K}}^x_{ni}(L)\mathbf{x}_{nt} = \underline{\mathbf{K}}^x_{ni}(L)\chi_{nt} + \underline{\mathbf{K}}^x_{ni}(L)\boldsymbol{\xi}_{nt}. \tag{21}$$

Let us show that the spectral density matrix of $\underline{\mathbf{K}}^x_{ni}(L)\boldsymbol{\xi}_{nt}$ tends to zero a.e. in $[-\pi,\ \pi]$. By (4) and Lemma 2,

$$|\mathbf{K}^x_{ni}(\theta)|^2 = \sum_{j=1}^{q} |p_{nj,i}(\theta)|^2$$

converges to zero a.e. in $[-\pi,\ \pi]$. Thus the result follows from Assumption 3 and

$$\mathbf{K}^x_{ni}(\theta)\mathbf{\Sigma}^{\xi}_n(\theta)\tilde{\mathbf{K}}^x_{ni}(\theta) \leq \lambda^{\xi}_{n1}(\theta)|\mathbf{K}^x_{ni}(\theta)|^2.$$

Moreover, since $|\mathbf{K}^x_{ni}(\theta)|^2 \leq q$ for any $\theta \in [-\pi,\ \pi]$, then the Lebesgue dominated convergence theorem applies and

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |\mathbf{K}^x_{ni}(\theta)|^2 d\theta = 0,$$

so that $\{\underline{\mathbf{K}}^x_{ni}(L),\ n \in \mathrm{N}\}$ is a DAS.

Now, from (20) and (21)

$$[\chi_{it} - \underline{\mathbf{K}}^x_{ni}(L)\chi_{nt}] + [\xi_{it} - \xi_{it,n}] = \underline{\mathbf{K}}^x_{ni}(L)\boldsymbol{\xi}_{nt}. \tag{22}$$

Consider the spectral density of the left-hand side of (22). Since ξ_{it} is orthogonal to all χ's at any leads and lags, this is the sum of the spectral densities of the two terms, minus two times the real part of the cross spectrum between $\xi_{it,n}$ and $[\chi_{it} - \underline{\mathbf{K}}^x_{ni}(L)\chi_{nt}]$. Consider firstly the cross spectrum between $\xi_{it,n}$ and χ_{it}. Setting $\mathbf{b}_i(L) = (b_{i1}(L),\ b_{i2}(L),\ \cdots, b_{iq}(L))$, we have

$$\chi_{it} = \mathbf{b}_i(L)(u_{1t},\ u_{2t},\ \cdots,\ u_{qt})' = \mathbf{b}_i(L)\tilde{\mathbf{A}}_n(L^{-1})(v_{n1t},\ v_{n2t},\ \cdots,\ v_{nqt})' + \mathbf{b}_i(L)\mathbf{S}_{nt},$$

where $(u_{1t},\ u_{2t},\ \cdots,\ u_{qt})' = \tilde{\mathbf{A}}_n(L^{-1})(v_{n1t},\ v_{n2t},\ \cdots,\ v_{nqt})' + \mathbf{S}_{nt}$ is the orthogonal decomposition analyzed in Lemma 3 and v_{njt} has the definition given in Lemma 4. Since $\chi_{it,n}$ is orthogonal to the terms $\underline{\mathbf{p}}^x_{nj}\mathbf{x}_{nt}$, for $j = 1,\ \ldots,\ q$, at any lead and lag, it also orthogonal at any lead and lag to the terms v_{njt}. Thus the cross spectrum between $\xi_{it,n}$ and χ_{it} is equal to the cross spectrum between $\xi_{it,n}$

and $\mathbf{b}_i(L)S_{nt}$. The squared modulus of the latter is bounded by the product of the spectral density of $\xi_{it,n}$, which is dominated by the spectral density of x_{it}, and the spectral density of $\mathbf{b}_i(L)\mathbf{S}_{nt}$, i.e., by

$$\mathbf{b}_i(e^{-i\theta})\mathbf{\Sigma}_n^S(\theta)\tilde{\mathbf{b}}_i(e^{i\theta}).$$

By Lemma 3 , all the entries of $\mathbf{\Sigma}_n^S(\theta)$ tend to zero a.e. in $[-\pi,\ \pi]$, so that the cross spectrum between $\xi_{it,n}$ and χ_{it} tends to zero a.e. in $[-\pi,\ \pi]$. Using the same argument, considering the cross spectrum between $\xi_{it,n}$ and $\underline{\mathbf{K}}^x_{ni}(L)\chi_{nt}$, we end up with the cross spectrum between $\xi_{it,n}$ and $\underline{\mathbf{K}}^x_{ni}(L)\mathbf{B}_n(L)\mathbf{S}_{nt}$, where $\mathbf{B}_n(L)$ is the $n \times q$ matrix having the vectors $\mathbf{b}_j(L)$, $j = 1, \dots, q$, on the rows. As for the spectral density of $\underline{\mathbf{K}}^x_{ni}(L)\mathbf{B}_n(L)\mathbf{S}_{nt}$, first observe that, since $\mathbf{\Sigma}^\chi_n(\theta) = \mathbf{B}_n(e^{-i\theta})\tilde{\mathbf{B}}_n(e^{i\theta})$ and $\mathbf{\Sigma}^x_n(\theta) = \mathbf{\Sigma}^\chi_n(\theta) + \mathbf{\Sigma}^\xi_n(\theta)$,

$$\begin{aligned}\mathbf{K}^x_{ni}(\theta)\mathbf{B}_n(e^{-i\theta})\tilde{\mathbf{B}}_n(e^{i\theta})\tilde{\mathbf{K}}^x_{ni}(\theta) = \mathbf{K}^x_{ni}(\theta)\mathbf{\Sigma}^\chi_n(\theta)\tilde{\mathbf{K}}^x_{ni}(\theta) &\le \mathbf{K}^x_{ni}(\theta)\mathbf{\Sigma}^x_n(\theta)\tilde{\mathbf{K}}^x_{ni}(\theta)\\ &= \sum_{j=1}^{q} |p^x_{nj,i}(\theta)|^2 \lambda^x_{nj}(\theta),\end{aligned}$$

which is bounded by the spectral density of x_{it} (see Lemma 2). Next, observe that the maximum eigenvalue of $\Sigma^S_n(\theta)$, which is a continuous function of the entries, tends to zero a.e. in $[-\pi,\ \pi]$.

As a consequence, since we have proved that the spectral density of the right hand side in (22) tends to zero a.e. in $[-\pi,\ \pi]$, so must the spectral densities of $\chi_{it} - \underline{\mathbf{K}}^x_{ni}(L)\chi_{nt}$ and $\xi_{it} - \xi_{it,n}$. Moreover, since both spectral densities are obviously bounded by integrable functions for any n, then, by the Lebesgue dominated convergence theorem the variance of both terms tend to zero, so that $\xi_{it,n}$ converges to ξ_{it} and $\chi_{it,n}$ to χ_{it}. This completes the proof of Proposition 3. QED

Let us now prove Proposition 4. Define

$$\mathcal{C}_n = \overline{\mathrm{span}}\left(\{\chi_{it},\ i = 1, \dots, n,\ t \in Z\}\right).$$

Obviously, $\mathcal{C}_n \subseteq \mathcal{U}$. We want to show that there exists a $\check{n}$ such that, for $n > \check{n}$, $\mathcal{C}_n = \mathcal{U}$. For, observe that the first q eigenvalues of $\mathbf{\Sigma}^\chi_n(\theta) = \mathbf{B}_n(e^{-i\theta})\tilde{\mathbf{B}}_n(e^{-i\theta})$ are equal to the eigenvalues of the $q \times q$ matrix $\tilde{\mathbf{B}}_n(e^{-i\theta})\mathbf{B}_n(e^{-i\theta})$ (see again Brillinger, 1981, Theorem 3.7.2, p. 72). Therefore, if $d_n(e^{-i\theta})$ is the determinant of $\tilde{\mathbf{B}}_n(e^{-i\theta})\mathbf{B}_n(e^{-i\theta})$, then

$$d_n(e^{-i\theta}) = \lambda^\chi_{n1}(\theta)\lambda^\chi_{n2}(\theta)\cdots\lambda^\chi_{nq}(\theta).$$

On the other hand, d_n is a rational function of $e^{-i\theta}$. Therefore, either d_n vanishes for any $\theta \in [-\pi,\ \pi]$, or for a subset of Lebesgue measure zero. But when d_n vanishes

everywhere in $[-\pi,\ \pi]$, the smallest eigenvalue of $\mathbf{\Sigma}^\chi_n$ vanishes everywhere in $[-\pi,\ \pi]$, and this, by Assumption 4, cannot occur for any n. Thus, there exists a $\check{n}$ such that for all $n > \check{n}$ the first q eigenvalues of $\mathbf{\Sigma}^\chi_n$ are positive, except for a subset of $[-\pi,\ \pi]$ ofmeasure zero. We have proved that, for $n > \check{n}$, the rank of the spectral density matrix of χ_{nt} is q a.e. in $[-\pi,\ \pi]$. As a consequence, $\mathcal{C}_n$ contains a q-dimensional orthonormal white noise (Rozanov, 1967, pp. 39-43), so that $\mathcal{C}_n = \mathcal{U}$. On the other hand, by Proposition 3, $\mathcal{C}_n \subseteq \mathcal{G}(\mathbf{x})$, so that $\mathcal{U} = \mathcal{G}(\mathbf{x})$ and Proposition 4 is proved. QED

PROOF OF PROPOSITION 5. For fixed T (and n), the T random variables $(\hat{\underline{\mathbf{K}}}^{Tt}_{ni}(L) - \underline{\mathbf{K}}_{ni}(L))\mathbf{x}_{nt}$, $t = 1, \dots, T$, are not identically distributed, due to two reasons: the truncation of the filters, which depends on t, and the *boundary effects*, which imply that the joint distributions of $(\hat{\underline{\mathbf{K}}}^{T}_{ni}(L), \mathbf{x}_{n1})$ and $(\hat{\underline{\mathbf{K}}}^{T}_{ni}(L), \mathbf{x}_{nT})$ are not the same as, e.g., those of $(\hat{\underline{\mathbf{K}}}^{T}_{ni}(L), \mathbf{x}_{n,T/2})$. However, the filters $\hat{\underline{\mathbf{K}}}^{T}_{ni}(L)$ and $\underline{\mathbf{K}}_{ni}(L)$ are both square-summable, and $\{\mathbf{x}_{nt};\ t \in Z\}$ is stationary, so that the influence of truncation is asymptotically negligible, as $T \to \infty$, for *central* values of T, i.e., for sequences of the form $t = t(T)$ such that

$$aT \le t(T) \le bT \qquad (23)$$

(automatically satisfying (11)).

Thus, for T large enough, the difference between $\mathrm{E}\left[|(\hat{\underline{\mathbf{K}}}^{Tt}_{ni}(L) - \underline{\mathbf{K}}_{ni}(L))\mathbf{x}_{nt}|\right]$ and $\mathrm{E}\left[|(\hat{\underline{\mathbf{K}}}^{T}_{ni}(L)) - (\underline{\mathbf{K}}_{ni}(L))\mathbf{x}_{nt}|\right]$ is arbitrarily small for any sequence $t = T(t)$ satisfying (23); note that expressions such as $\underline{\mathbf{K}}_{ni}(L)\mathbf{x}_{nt}$ typically involve the complete, non observed, process $\{\mathbf{x}_{nt};\ t \in Z\}$. Similarly, the *boundary effects* affecting the joint distribution of $\mathbf{x}_{nt}$ and $\hat{\underline{\mathbf{K}}}^{T}_{ni}(L)$ are asymptotically nil as $T \to \infty$.

Piecing these two facts together, for all $\eta > 0$, there exists a $T_2 = T_2(n, \eta)$ such that, for all sequences $t_1(T)$ and $t_2(T)$ satisfying (23), and all $T \ge T_2$,

$$\begin{aligned}\Bigg|\mathrm{E}&\left[|(\hat{\underline{\mathbf{K}}}^{Tt_1}_{ni}(L) - \underline{\mathbf{K}}_{ni}(L))\mathbf{x}_{nt_1}| \,\Big|\, \sup_{\theta\in[-\pi,\pi]} |\hat{\mathbf{K}}^{T}_{ni}(\theta) - \mathbf{K}_{ni}(\theta)| \le \delta\right]\\ &- \mathrm{E}\left[|(\hat{\underline{\mathbf{K}}}^{Tt_2}_{ni}(L) - \underline{\mathbf{K}}_{ni}(L))\mathbf{x}_{nt_2}| \,\Big|\, \sup_{\theta\in[-\pi,\pi]} |\hat{\mathbf{K}}^{T}_{ni}(\theta) - \mathbf{K}_{ni}(\theta)| \le \delta\right]\Bigg| \le \eta.\end{aligned}$$

It follows that, for $T \ge T_2(n, \frac{\epsilon^2}{4}\frac{\eta}{8})$,

$$\mathrm{E}\left[|(\hat{\underline{\mathbf{K}}}_{ni}^{Tt}(L) - \hat{\underline{\mathbf{K}}}_{ni}(L))\mathbf{x}_{nt}|\Big| \sup_{\theta\in[-\pi,\pi]} |\hat{\mathbf{K}}_{ni}^{T}(\theta) - \mathbf{K}_{ni}(\theta)| \leq \delta\right]$$

$$\leq \mathrm{E}\left[(bT-aT)^{-1}\Sigma_{t=aT}^{bT}|(\hat{\underline{\mathbf{K}}}_{ni}^{T}(L) - \underline{\mathbf{K}}_{ni}(L))\mathbf{x}_{nt}|\Big| \sup_{\theta\in[-\pi,\pi]} |\hat{\mathbf{K}}_{ni}^{T}(\theta) - \mathbf{K}_{ni}(\theta)| \leq \delta\right]$$
$$+\frac{\epsilon^2\eta}{32},$$

where $\Sigma_{t=aT}^{bT}$ stands for a sum over t running from the smallest integer $\lfloor aT \rfloor$ larger than or equal to aT to the largest integer $\lceil bT \rceil$ smaller than or equal to bT (a window of width $(\lceil bT \rceil - \lfloor aT \rfloor)$. For simplicity, we write $(bT-aT)^{-1}$ for $(\lceil bT \rceil - \lfloor aT \rfloor)^{-1}$.

Hence,

$$R_{1t}^{nT} \leq \frac{4}{\epsilon^2}\mathrm{E}\left[\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(\hat{\underline{\mathbf{K}}}_{ni,k}^{T} - \underline{\mathbf{K}}_{ni,k})(bT-aT)^{-1}\right.$$
$$\left.\sum_{t=aT}^{bT}\mathbf{x}_{n,t-k}\mathbf{x}'_{n,t-l}(\hat{\underline{\mathbf{K}}}_{ni,l}^{T} - \underline{\mathbf{K}}_{ni,l})'\Big| \sup_{\theta\in[-\pi,\pi]} |\hat{\mathbf{K}}_{ni}^{T}(\theta) - \mathbf{K}_{ni}(\theta)| \leq \delta\right]$$
$$+\frac{\eta}{8}+\frac{\eta}{4}$$
$$\leq \frac{4\delta^2}{\epsilon^2}\mathrm{E}\left[\int_{-\pi}^{\pi}\overline{\lambda}_{n1}^{T}(\theta)d\theta\right] + \frac{3\eta}{8},$$

where $\overline{\lambda}_{n1}^{T}(\theta)$ denotes the first dynamic eigenvalue associated whith the *pseudo-empirical* cross-covariance function $\overline{\mathbf{\Gamma}}_{k,l}^{nT} = (bT-aT)^{-1}\sum_{t=aT}^{bT}(\mathbf{x}_{n,t-k}\mathbf{x}'_{n,t-l})$. In this *pseudo-empirical* cross-covariance structure, each cross-covariance matrix is estimated on the basis of a window of lenght $(bT-aT)$. Moreover, all covariances are "estimated" using the same number of "observations"; as a consequence, we can apply the argument used by Grenander and Rosenblatt, 1957 p. 262 to show that, under the assumption of linearity of the observed process, we have convergence (in the mean square) of the corresponding estimated spectral density $\overline{f}_{n}^{T}(\theta)$ to its theoretical counterpart, uniformly in $[-\pi,\ \pi]$, as $T \to \infty$.

Since eigenvalues and the components of eigenvectors are continuous functions of the entries of the corresponding spectral densities, $\int_{-\pi}^{\pi}\overline{\lambda}_{n1}^{T}(\theta)d\theta$ converges in probability, as $T \to \infty$, to $\int_{-\pi}^{\pi}\lambda_{n1}(\theta)d\theta$. Moreover, the sequence $\int_{-\pi}^{\pi}\overline{\lambda}_{n1}^{T}(\theta)d\theta$ is bounded by $\int_{-\pi}^{\pi}\mathrm{trace}[\overline{f}_{n}^{T}(\theta)]d\theta = \mathrm{trace}[\overline{\mathbf{\Gamma}}_{00}^{nT}] = (bT-aT)^{-1}\sum_{t=aT}^{bT}\sum_{i=1}^{n}x_{n,t,i}^2$. This latter

quantity, as an empirical variance, under fourth-order moments assumptions, has uniformly bounded moment of order two. It follows (cf. Billingsley, 1995, p. 338) that $\int_{-\pi}^{\pi}\overline{\lambda}_{n1}^{T}(\theta)d\theta$ is uniformly integrable (as $T \to \infty$), so that

$$\lim_{T\to\infty}\mathrm{E}\left[\int_{-\pi}^{\pi}\overline{\lambda}_{n1}^{T}(\theta)d\theta\right] = \mathrm{E}\left[\int_{-\pi}^{\pi}\lambda_{n1}^{T}(\theta)d\theta\right] = \int_{-\pi}^{\pi}\lambda_{n1}^{T}(\theta)d\theta.$$

Thus, there exists some $T_3(n)$ such that, for all $T \geq T_3$,

$$\mathrm{E}\left[\int_{-\pi}^{\pi}\overline{\lambda}_{n1}^{T}(\theta)d\theta\right] \leq 2\int_{-\pi}^{\pi}\lambda_{n1}^{T}(\theta)d\theta.$$

Summing up, for $n \geq N_0(\epsilon,\eta)$ and $T \geq T_0$, with $T_0 = T_0(n,\epsilon,\eta) = \max\left(T_1(n,\delta,\frac{\eta}{4}), T_2(n,\frac{\epsilon^2\eta}{32}), T_3(n,\eta), T_3(n)\right)$ where $\delta^2 = \frac{\epsilon^2\eta}{128\int_{-\pi}^{\pi}\lambda_{n1}(\theta)d\theta}$, we have proved that $R_{1t}^{nt} \leq \frac{\eta}{2}$. The proposition follows. QED

1. Maria Cristina Marcuzzo [1985] "Yoan Violet Robinson (1903-1983)", pp. 134

2. Sergio Lugaresi [1986] "Le imposte nelle teorie del sovrappiù", pp. 26

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133. Carlo Alberto Magni [1996] "Un esempio di investimento industriale con interazione competitiva e avversione al rischio" pp. 20

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151. Enrico Giovannetti [1996] "Il I° ciclo del Diploma Universitario Economia e Amministrazione delle Imprese", pp. 25

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