The generalized dynamic factor model: representation theory

by

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ABSTRACT

This paper, along with the companion paper Forni, Hallin, Lippi and Reichlin (1998), introduces a new model—the *generalized dynamic factor model*—for the empirical analysis of financial and macroeconomic data sets characterized by a large number of observations both cross-section and over time. This model provides a generalization of the static approximate factor model of Chamberlain (1983) and Chamberlain and Rothschild (1983) by allowing serial correlation within and across individual processes, and of the dynamic factor model (or index model) of Sargent and Sims (1977) and Geweke (1977) by allowing for non-orthogonal idiosyncratic terms. While the companion paper concentrates on identification and estimation, here we give a full characterization of the generalized dynamic factor model in terms of observable spectral density matrices, thus laying a firm basis for the empirical implementation of the model.

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1. Introduction[1]

1.1 Data sets with many data points both over time and across sections are becoming increasingly available. Think for instance of macroeconomic series on output or employment which are observed for a large number of countries, regions or sectors, or of financial time series such as the returns on many different assets. Such data sets typically present a good deal of regularity along the time dimension, so that each time series, taken in isolation, can be successfully handled by using standard stationary models or their extensions. By contrast, along the cross sectional dimension, data do not have a natural ordering and correlations do not present any regular structure. Yet, the series are strongly dependent on each other, implying that univariate modeling would waste information.

We do not have a satisfactory theoretical framework for extracting and analyzing the enormous amount of information embedded in such large cross sections of time series. VAR models would be suitable for a small subset of time series, but are inadequate for the whole data set, because of the huge number of parameters to estimate. The dynamic factor analytic or index model (Sargent and Sims, 1977, Geweke, 1977) is much better suited, since it is both flexible and parsimonious: each variable is represented as the sum of a common component—i.e. a term depending, possibly with heterogeneous dynamic responses, on a small number of unobserved factors which are common to all variables—and an idiosyncratic component, which is orthogonal at any lead and lag both to the common factors and to the idiosyncratic components of all the other variables.

This feature, mutual orthogonality of the idiosyncratic components at any lead and lag, represents a serious weakness of the index model. The assumption is necessary for identification, but is severely restrictive. As a first example, consider the output of different industries linked to each other by input-output relations. The output of sector A may well be related to the output of sector B in a way which is intimately 'cross-regressive', so that an idiosyncratic shock originated in B propagates, possibly with a lag, to sector A. Similar local interactions can also arise when there are 'intermediate' shocks, i.e. shocks which are neither common nor strictly idiosyncratic, such as local events affecting directly more than one area or technological shocks affecting a few sectors. Finally, consider a data set including both employment and income for many regions, and assume that each variable is driven by a national and a regional shock, the second being orthogonal to the first. The regional components of employment and income, while being orthogonal for different regions, are likely to be correlated for the same region. In such a case, although employment, or income, taken in isolation would satisfy the orthogonality assumption, the index model could not be used to handle the whole data set.

In this paper, and in the companion paper Forni, Hallin, Lippi and Reichlin (1998), a new model, that we will call the *generalized dynamic factor model*, is introduced and analyzed. The model has three important features: (1) it is a finite dynamic factor model, i.e. the variables depend on a finite number of factors with a quite general lag structure; (2) it is based on an infinite sequence of variables and is therefore specifically designed for the analysis of large cross sections of time series; (3) it allows for both contemporaneous and lagged correlation between the idiosyncratic terms, and is therefore more general than the traditional index model.

1.2 Let us briefly summarize the results of the paper. In Section 2 we give our basic definitions and assumptions. We start with a double sequence of stochastic variables $\{x_{it}, \ i \in \mathbb{N}, \ t \in \mathbb{Z}\}$. We assume that $\{x_{it}, \ t \in \mathbb{Z}\}$ is stationary for any i and costationary with $\{x_{it}, \ t \in \mathbb{Z}\}$ for any j. We do not assume an ARMA structure for the x's. We only require the existence of a spectral density matrix Σ_n^x for the vector $(\, x_{1t} \quad x_{2t} \quad \cdots \quad x_{nt} \,)'$.

In Section 3 we introduce idiosyncratic sequences. To give a simple illustration of the definition of idiosyncratic sequences adopted here, let us consider a sequence $\{y_i, \ i \in \mathbb{N}\}$ of mutually orthogonal variables, such that $\text{var}(y_i) = \sigma^2$. Taking a sequence of averages $Y_n = \sum_{i=1}^n a_{ni} y_i$, the variance $\text{var}\,(Y_n) = \sigma^2 \sum_{i=1}^n a_{ni}^2$ tends to zero if and only if $\sum_{i=1}^n a_{ni}^2$ tends to zero; this occurs typically with the arithmetic mean, $a_{ni} = 1/n$. Now, the property of a vanishing variance for sequences of averages whose squared weights tend to zero does not require that the y's be mutually orthogonal: for example, if y_i and y_j are correlated with the correlation declining as $|i - j|$, then the $\text{var}(Y_n)$ vanishes asymptotically. This vanishing variance of averages, not orthogonality, is precisely what we need in our construction. Thus, in our definition, the sequence of the x's is idiosyncratic if convergence to zero occurs for any weighted average, both cross-section and over time,

$$\sum_{i=1}^n \sum_{h=-k}^k a_{nih} x_{it-h},$$

provided that the sum of the squared weights tends to zero. We prove, Theorem 1, that x_{it} is idiosyncratic if and only if the maximum eigenvalue of Σ_n^x is dominated by an essentially bounded function of the frequency θ.

In Section 4 we introduce our generalized dynamic factor model, i.e. a sequence $\{x_{it}, \ i \in \mathbb{N}, \ t \in \mathbb{Z}\}$ such that

$$x_{it} = b_{i1}(L)u_{1t} + b_{i2}(L)u_{2t} + \cdots + b_{iq}(L)u_{qt} + \xi_{it},$$

where $b_{ij}(L)$ is a square-summable filter, $(\, u_{1t} \quad u_{2t} \quad \cdots \quad u_{qt} \,)'$ is an orthonormal vector white noise, ξ_{it} is idiosyncratic and orthogonal to the u's at any lead and lag, with the filters $b_{ij}(L)$ fulfilling a condition ensuring that no representation with a smaller number of "common factors" is possible. We prove in Theorem 2 that a sequence has a generalized dynamic factor structure with q factors if and only if: (I) the $(q + 1)$-th eigenvalue of Σ_n^x, in decreasing order, is dominated for any n by an essentially bounded function of the frequency θ; (II) as n tends to infinity, the q-th eigenvalue diverges for θ almost everywhere in $[-\pi, \ \pi]$.

Thus the unobservable factor structure is completely characterized in terms of properties of the observable matrices Σ_n^x. This result has a very important consequence for empirical analysis, as it provides the theoretical basis for heuristic criteria or formal tests in which the sequence of nested matrices Σ_n^x is employed to determine whether the model has a finite dynamic factor structure and what is the number of factors. More precisely, evidence in favor of conditions (I) and (II), with the eigenvalues computed from estimated spectral density matrices, can be interpreted, given the "if" part of Theorem 2, as evidence that, firstly, the variables follow a generalized dynamic factor model, and, secondly, that the number of factors is q.

Theorems 3 and 4 establish uniqueness of the idiosyncratic component ξ_{it} and of the common component $x_{it} - \xi_{it}$. It must be pointed out that this identification result holds for the whole infinite sequence of the variables x_{it}, not for its finite subsets: otherwise stated,

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identification occurs in the limit, when the size of the cross-section tends to infinity. Moreover, note that identification of χ_{it} does not imply identification of the u's or of the filters $b_{ij}(L)$, that might be achieved only by imposing further, economically motivated, restrictions. Such an issue will not be discussed in this paper. Finally, in Theorem 5 we show that the common component of x_{it} can be recovered as the limit of the projection of x_{it} on the dynamic principal components. This result provides a firm basis for estimation theory.

The case in which the x's are either difference or trend stationary is shortly discussed in Section 5. Finally, some results that appear as common knowledge but for which no easy reference was available are gathered in the Appendix.

1.3 In spite of some inevitable overlapping this paper is complementary with respect to the companion paper mentioned above. The latter concentrates on identification and estimation of the common and idiosyncratic components, while the main aim of this paper is the full characterization given in Theorem 2. Moreover, the assumption of rational spectral density, made in the companion paper, is dropped completely here.

Correlated idiosyncratic factors, along with infinite cross sectional size, have been introduced in a static model for asset markets by Chamberlain (1983) and Chamberlain and Rothschild (1983). Our Theorem 2 is a generalization to stochastic processes of results proved in the static case by Chamberlain and Rothschild. Related models can also be found in Quah and Sargent (1993), Forni and Reichlin, (1996, 1998), Forni and Lippi (1997), Stock and Watson (1998).

2. Notation, Basic Definitions and Lemmas

Given a complex matrix $\mathbf{D}$, we denote by $\tilde{\mathbf{D}}$ the complex conjugate of the transpose of $\mathbf{D}$. Inner product and norm in $\mathbb{C}^s$ are the usual Euclidean entities $(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^s v_i \bar{w}_i$ and $|\mathbf{v}| = \sqrt{\sum_{i=1}^s |v_i|^2}$ respectively.

Let $\mathcal{P} = (\Omega, \mathcal{F}, P)$ be a probability space and let $L_2(\mathcal{P}, \mathbb{C})$ be the linear space of all complex-valued, zero-mean, square-integrable random variables defined on Ω. We recall that $L_2(\mathcal{P}, \mathbb{C})$, with the inner product defined as $\langle x, y \rangle = \mathrm{E}(x\bar{y}) = \mathrm{cov}(x, y)$, and the norm as $||x|| = \sqrt{\mathrm{E}(|x|^2)} = \sqrt{\mathrm{var}(x)}$, is a Hilbert space on the complex field $\mathbb{C}$. If Q is a subset of $L_2(\mathcal{P}, \mathbb{C})$ we shall denote by $\overline{\mathrm{span}}(Q)$ the minimum closed linear subspace of $L_2(\mathcal{P}, \mathbb{C})$ containing Q. If V is a closed linear subspace of $L_2(\mathcal{P}, \mathbb{C})$ and $x \in L_2(\mathcal{P}, \mathbb{C})$, we denote by $\mathrm{proj}(x|V)$ the orthogonal projection of x on V.

Now consider a double sequence

$$\mathbf{x} = \{x_{it}, \ i \in \mathbb{N}, \ t \in \mathbb{Z}\},$$

where $x_{it} \in L_2(\mathcal{P}, \mathbb{C})$. We adopt the following notation: $\mathbf{X} = \overline{\mathrm{span}}(\mathbf{x})$; $\mathbf{x}_{nt}$ is the n-dimensional **column** vector $(\, x_{1t} \quad x_{2t} \quad \cdots \quad x_{nt} \,)'$. Often, when no confusion can arise, we speak of the process z_t, meaning the process $\{z_t, \ t \in \mathbb{Z}\}$. We also speak of the spectral density of z_t. Moreover, considering the m-dimensional vector process

$$\mathbf{y} = \{(\, y_{1t} \quad y_{2t} \quad \cdots \quad y_{mt} \,)', \ t \in \mathbb{Z}\},$$

we say that $\mathbf{y}$ belongs to $\mathbf{Y} \subseteq L_2(\mathcal{P}, \mathbb{C})$ if y_{jt} belongs to $\mathbf{Y}$ for any j and t. In the same way, we use $\overline{\mathrm{span}}(\mathbf{y})$ to indicate $\overline{\mathrm{span}}(\{y_{jt}, \ j = 1, \ldots, m, \ t \in \mathbb{Z}\})$.

Assumption 1. *For any $n \in \mathbb{N}$: (1) the process $\mathbf{x}_{nt}$ is covariance stationary; (2) the spectral measure of $\mathbf{x}_{nt}$ is absolutely continuous (with respect to the Lebesgue measure on $[-\pi, \ \pi]$), i.e. $\mathbf{x}_{nt}$ has a spectral density (see Rozanov, 1967, pp. 19-20).*

We denote by Σ_n^x the spectral density matrix of $\mathbf{x}_{nt}$ and recall that Σ_n^x is Hermitian, non-negative definite for any $\theta \in [-\pi, \pi]$, Lebesgue-measurable, and that $\int_{-\pi}^{\pi} \Sigma_n^x(\theta)d\theta$ is equal to the variance-covariance matrix of $\mathbf{x}_{nt}$.

Remark 1. Given y and z in $\mathbf{X}$, by definition $||y - z||^2 = \mathrm{var}(y - z)$. On the other hand, we can define in a natural way two stochastic processes y_t and z_t such that $||y - z||^2 = ||y_t - z_t||^2 = \int_{-\pi}^{\pi} f(\theta)d\theta$, where f is the spectral density of $y_t - z_t$. For, the definition of $\mathbf{X}$ implies the existence of square-summable filters $a_{nj}(L)$ and $b_{nj}(L)$ such that

$$y = \lim_n \sum_{j=1}^n a_{nj}(L)x_{j0}, \qquad z = \lim_n \sum_{j=1}^n b_{nj}(L)x_{j0}.$$

Defining

$$y_t = \lim_n \sum_{j=1}^n a_{nj}(L)x_{jt}, \qquad z_t = \lim_n \sum_{j=1}^n b_{nj}(L)x_{jt},$$

the processes y_t and z_t are stationary and costationary with $\mathbf{x}_{nt}$ for any $n \in \mathbb{N}$. Moreover, $y = y_0$, $z = z_0$ and $||y - z||^2 = \int_{-\pi}^{\pi} f(\theta)d\theta$, where f is the spectral density of $y_t - z_t$ (for the existence of the spectral density of y_t, z_t and $y_t - z_t$, see the Appendix, Fact S). With the above argument in mind, the generic element of $\mathbf{X}$ will be often referred to as y_t, z_t, etc., rather than y, z, etc., where y_t, z_t, etc. are stationary and costationary with $\mathbf{x}_{nt}$, for any n. Identical considerations hold if we consider a vector $\mathbf{y}$ belonging to $\mathbf{X}$.

Definition 1. For $i = 1, \ldots, n$ let

$$\lambda_{ni}^x : [-\pi, \pi] \mapsto \mathbb{R}$$

be defined as the function associating with $\theta \in [-\pi, \pi]$ the i-th eigenvalue, in descending order, of $\Sigma_n^x(\theta)$. The functions λ_{ni}^x will be called the **dynamic eigenvalues** of Σ_n^x.

In the Appendix we prove the following lemmas.

Lemma 1. The functions λ_{ni}^x are Lebesgue-measurable in $[-\pi, \pi]$ for any $n \in \mathbb{N}$ and $i \leq n$ (in Remark 3 below we show that λ_{ni}^x has finite integral).

Lemma 2. Given i, for $n \geq i$, $\lambda_{ni}^x(\theta)$ is non-decreasing as a function of n for any $\theta \in [-\pi, \pi]$, i.e. $\lambda_{ni}^x(\theta) \leq \lambda_{n+1i}^x(\theta)$.

A consequence of Lemma 2 is that $\lim_n \lambda_{ni}^x(\theta)$ exists for any i and θ, and equals $\sup_n \lambda_{ni}^x(\theta)$.

Definition 2. For any i we define the function λ_i^x by

$$\lambda_i^x(\theta) = \sup_n \lambda_{ni}^x(\theta).$$

It must be pointed out that λ_i^x is an extended real function, i.e. its value may be infinite. Note also that $\{\theta : \lambda_i^x(\theta) = \infty\}$ may be of null or positive measure, and even coincide with $[-\pi, \pi]$.

Now consider the space $L_2^n([-\pi, \pi], \mathbb{C})$ of all functions $\mathbf{f} : [-\pi, \pi] \mapsto \mathbb{C}^n$, with

$$\mathbf{f}(\theta) = (\, f_1(\theta) \quad f_2(\theta) \quad \cdots \quad f_n(\theta) \,)$$

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(a row vector) and $f_i : [-\pi, \pi] \mapsto \mathbb{C}$, such that

$$\int_{-\pi}^{\pi} |\mathbf{f}(\theta)|^2 d\theta = \int_{-\pi}^{\pi} \sum_{i=1}^{n} |f_i(\theta)|^2 d\theta < \infty.$$

With the inner product of $\mathbf{f}$ and $\mathbf{g}$ and the norm of $\mathbf{f}$ defined respectively as

$$\langle \mathbf{f}, \mathbf{g} \rangle = (2\pi)^{-1} \int_{-\pi}^{\pi} \sum_{i=1}^{n} f_i(\theta) \bar{g}_i(\theta) d\theta \quad \text{and} \quad ||\mathbf{f}|| = \sqrt{(2\pi)^{-1} \int_{-\pi}^{\pi} \sum_{i=1}^{n} |f_i(\theta)|^2 d\theta},$$

$L_2^n([-\pi, \pi], \mathbb{C})$ is a Hilbert space. In the Appendix we prove the following results.

Lemma 3. *There exist n functions $\mathbf{p}_{ni}^x$, $i = 1, \ldots, n$, belonging to $L_2^n([-\pi, \pi], \mathbb{C})$, such that*
(1) $|\mathbf{p}_{ni}^x(\theta)| = 1$, for any $\theta \in [-\pi, \pi]$;
(2) $\mathbf{p}_{ni}^x(\theta) \tilde{\mathbf{p}}_{nj}^x(\theta) = 0$, for $i \neq j$ and any $\theta \in [-\pi, \pi]$;
(3) $\mathbf{p}_{ni}^x(\theta) \mathbf{\Sigma}_n^x(\theta) = \lambda_{ni}^x(\theta) \mathbf{p}_{ni}^x(\theta)$ for any $\theta \in [-\pi, \pi]$.

Definition 3. *An n-tuple of functions $\mathbf{p}_{ni}^x$ fulfilling (1), (2) and (3) of Lemma 3 will be called a set of **dynamic eigenvectors**.*[2]

Given $\mathbf{f} \in L_2^n([-\pi, \pi], \mathbb{C})$, consider the Fourier expansion

$$\mathbf{f}(\theta) = \sum_{k=-\infty}^{\infty} \mathbf{F}_k e^{-ik\theta},$$

where $\mathbf{F}_k = \int_{-\pi}^{\pi} \mathbf{f}(\theta) e^{ik\theta} d\theta \in \mathbb{C}^n$. We shall indicate by $\underline{f}(L)$ the square summable n-dimensional filter

$$\sum_{k=-\infty}^{\infty} \mathbf{F}_k^{\mathbf{f}} L^k.$$

We have

$$\sum_{k=-\infty}^{\infty} |\mathbf{F}_k|^2 = (2\pi)^{-1} \int_{-\pi}^{\pi} |\mathbf{f}(\theta)|^2 d\theta = ||\mathbf{f}||^2.$$

In particular, $\underline{\mathbf{p}}_{nj}^x(L)$ is the filter associated with the dynamic eigenvector $\mathbf{p}_{nj}^x : [-\pi, \pi] \to \mathbb{C}^n$. Conversely, if

$$\mathbf{A}(L) = \sum_{k=-\infty}^{\infty} \mathbf{A}_k L^k$$

is a square summable n-dimensional filter, consider the function $\mathbf{A}^\circ$ defined as

$$\mathbf{A}^\circ(\theta) = \mathbf{A}(e^{-i\theta}) = \sum_{k=-\infty}^{\infty} \mathbf{A}_k e^{-ik\theta}.$$

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[2] We use "dynamic" for eigenvectors and eigenvalues of $\mathbf{\Sigma}_n^x$ and related matrices to insist on the difference between the dynamic analysis developed here and the static approach, based on the eigenvalues of variance-covariance matrices. For a general treatment of eigenvalues and eigenvectors of spectral density matrices, and related filters, see Brillinger (1981).

Obviously $\mathbf{A}^\circ$ belongs to $L_2^n([-\pi,\ \pi],\mathbb{C})$. Moreover $\underline{\mathbf{A}}^\circ(L) = \mathbf{A}(L)$.

Definition 4. *If the functions* $\mathbf{p}_{nj}^x$, $j = 1,\dots,n$ *form a set of dynamic eigenvectors, then* $\underline{\mathbf{p}}_{nj}^x(L)\mathbf{x}_{nt}$, $j = 1,\dots,n$ *is a set of* **dynamic principal components** *of* x_{nt}.

The main results presented below will employ the filters $\underline{\mathbf{p}}_{ni}^x(L)$, derived from the dynamic eigenvectors. In general, being obtained from the Fourier expansion of expressions involving polynomial roots, such filters are bilateral and infinite, even when the vector process $\mathbf{x}_{nt}$ is a finite moving average.

3. Dynamic averaging sequences, aggregation space, idiosyncratic variables

Definition 5. *Consider a sequence of positive integers* $\{s_n,\ n \in \mathbb{N}\}$ *and a sequence*

$$\{\mathbf{a}_n(L),\ n \in \mathbb{N}\},$$

where

$$\mathbf{a}_n^\circ = (\ a_{n1}^\circ \quad a_{n2}^\circ \quad \cdots \quad a_{ns_n}^\circ\ ) \in L_2^{s_n}([-\pi,\ \pi],\mathbb{C}).$$

We say that $\{\mathbf{a}_n(L),\ n \in \mathbb{N}\}$ *is a* **dynamic averaging sequence**, *DAS henceforth, if*

$$\lim_n ||\mathbf{a}_n{}^\circ||^2 = \lim_n \int_{-\pi}^\pi |\mathbf{a}_n^\circ(\theta)|^2 d\theta = 0.$$

Definition 6. *Let* $y_t \in \mathbf{X}$. *We say that* y_t *is an* **aggregate** *if there exists a DAS* $\{\mathbf{a}_n(L),\ n \in \mathbb{N}\}$ *such that*

$$\lim_n \mathbf{a}_n(L)\mathbf{x}_{s_nt} = y_t.$$

The set of all the aggregates will be denoted by $\mathcal{G}(\mathbf{x})$ *and called the* **aggregation subspace** *of* $\mathbf{X}$.

Remark 2. In Assumption 1 and in Definition 5 we have not supposed that the entries of $\mathbf{\Sigma}_n^x$ or of $\mathbf{a}_n^\circ$ are bounded in $[-\pi,\ \pi]$. As a consequence, the elements of the sequence $\mathbf{a}_n(L)\mathbf{x}_{s_nt}$ does not necessarily have finite variance, i.e. the integrals $\int_{-\pi}^\pi \mathbf{a}_n^\circ(\theta)\mathbf{\Sigma}_{s_n}^x(\theta)\tilde{\mathbf{a}}_n^\circ(\theta)d\theta$ are not necessarily finite, which means that $\mathbf{a}_n(L)\mathbf{x}_{s_nt}$ is not necessarily an element of $L_2(\mathcal{P},\mathbb{C})$. However, convergence of $\{\mathbf{a}_n(L)\mathbf{x}_{s_nt},\ n \in \mathbb{N}\}$ has an obvious definition: $\{\mathbf{a}_n(L)\mathbf{x}_{s_nt},\ n \in \mathbb{N}\}$ converges if $\mathbf{a}_n(L)\mathbf{x}_{s_nt}$ has finite variance for n greater than some n_1 and if the sequence $\{\mathbf{a}_n(L)\mathbf{x}_{s_nt},\ n \in \mathbb{N},\ n > n_1\}$ converges.

Remark 3. Dynamic averaging of $\mathbf{x}$, according to Definition 5, is nothing other than averaging simultaneously both in the cross-section and the time dimension. Precisely

$$\mathbf{a}_n(L)\mathbf{x}_{s_nt} = \sum_{i=1}^{s_n} a_{ni}(L)x_{it} = \sum_{i=1}^{s_n}\sum_{k=-\infty}^\infty a_{nik}x_{it-k},$$

where the coefficients a_{nik} are complex numbers fulfilling the condition

$$\lim_n \sum_{i=1}^{s_n}\sum_{k=-\infty}^\infty |a_{nik}|^2 = 0.$$

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Infinite averaging in the time dimension is convenient since, as anticipated above, the averaging sequences that we are going to employ are typically infinite, but not strictly necessary. It is easily seen that the same aggregation space would be obtained by taking finite averages of any length in the time dimension

$$\sum_{i=1}^{s_n} \sum_{k=-\rho_{ni}}^{\tau_{ni}} a_{nik} x_{it-k},$$

with the condition

$$\lim_n \sum_{i=1}^{s_n} \sum_{k=-\rho_{ni}}^{\tau_{ni}} |a_{nik}|^2 = 0.$$

Lemma 4. *The set* $\mathcal{G}(\mathbf{x})$ *is a closed subspace of* $\mathbf{X}$.
Proof. Assume that $z_t = \lim_m y_{mt}$, with $y_{mt} \in \mathcal{G}(\mathbf{x})$. Let $y_{mt} = \lim_n \mathbf{a}_{mn}(L) \mathbf{x}_{s_{mn}t}$, where

$$\{\mathbf{a}_{mn}(L),\ n \in \mathbb{N}\}$$

is a DAS. Let m_i be such that $||z_t - y_{m_it}|| < 1/i$ and n_i such that $||\mathbf{a}_{m_in_i}^o|| < 1/i$ and $||y_{m_it} - \mathbf{a}_{m_in_i}(L)\mathbf{x}_{s_{m_in_i}t}|| < 1/i$. The sequence

$$\begin{pmatrix} \mathbf{a}_{m_1n_1}(L) & \mathbf{a}_{m_2n_2}(L) & \cdots & \end{pmatrix}$$

is a DAS and

$$||z_t - \mathbf{a}_{m_in_i}(L)\mathbf{x}_{s_{m_in_i}t}|| \le ||z_t - y_{m_it}|| + ||y_{m_it} - \mathbf{a}_{m_in_i}(L)\mathbf{x}_{s_{m_in_i}t}|| < 2/i.$$

QED

Definition 7. *Suppose that* $\mathbf{x}$ *fulfills Assumption 1. We say that* $\mathbf{x}$ *is* **idiosyncratic** *if* $\lim_n \mathbf{a}_n(L)\mathbf{x}_{s_nt} = 0$ *for any DAS* $\{\mathbf{a}_n(L),\ n \in \mathbb{N}\}$.

If $\mathbf{x}$ is idiosyncratic then obviously $\mathcal{G}(\mathbf{x}) = \{0\}$. However, as the next example shows, the converse does not hold.

Example 1. Assume that $x_{it} \perp x_{jt-k}$ for any $i \ne j$ and any $k \in \mathbb{Z}$, that x_{it} is a white noise for any i, and that $||x_{it}||^2 = i$. Define

$$\mathbf{c}_n(L) = \frac{1}{\sqrt{i}} \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The sequence $\{\mathbf{c}_n(L),\ n \in \mathbb{N}\}$ is a DAS. Moreover $||\mathbf{c}_n(L)\mathbf{x}_{nt}||^2 = 1$, so that $\mathbf{x}$ is not idiosyncratic. Now let y_t be an aggregate, so that

$$y_t = \lim_n \mathbf{a}_n(L)\mathbf{x}_{s_nt} = \lim_n \sum_{j=1}^{s_n} a_{nj}(L)x_{jt} = \lim_n \sum_{j=1}^{s_n} \sum_{k=-\infty}^{\infty} a_{njk} x_{jt-k}, \tag{1}$$

where $\{\mathbf{a}_n(L),\ n \in \mathbb{N}\}$ is a DAS. Since $y_t \in \mathbf{X}$ and the x_{it}'s are mutually orthogonal white noises, then

$$y_t = \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} b_{jk} x_{it-k}. \tag{2}$$

Moreover, representations (1) and (2) are unique and $\lim_n a_{njk} = b_{jk}$ for any j and k. On the other hand,

$$\lim_n \int_{-\pi}^{\pi} |\mathbf{a}_n^{\circ}(\theta)|^2 d\theta = \lim_n \left[\sum_{j=1}^{s_n} \sum_{k=-\infty}^{\infty} |a_{njk}|^2 \right] = 0,$$

so that $b_{jk} = 0$ for any j and k, i.e. $y_t = 0$. Thus $\mathcal{G}(\mathbf{x}) = \{0\}$ although $\mathbf{x}$ is not idiosyncratic.

If the vector $\mathbf{x}_{nt}$ is a white noise for any n, i.e. if the matrix Σ_n^x and its eigenvalues are constant as functions of θ, then $\mathbf{x}$ is idiosyncratic if and only if λ_{n1}^x is bounded as a function of n (see Chamberlain, 1983, Chamberlain and Rothschild, 1983). The theorem below generalizes this result to any $\mathbf{x}$ fulfilling Assumption 1: $\mathbf{x}$ is idiosyncratic if and only if the functions λ_{n1}^x are uniformly bounded in $[-\pi, \pi] - D$, where D has null Lebesgue measure, i.e. if there exists a real M such that $\lambda_{n1}^x(\theta) \leq M$ for any $n \in \mathbb{N}$ and $\theta \in [-\pi, \pi] - D$.

In the sequel $\mathcal{L}$ will denote the Lebesgue measure on $\mathbb{R}$. Let us recall that an extended real function f is **essentially bounded** if there exists a real c and a subset D of $[-\pi, \pi]$ such that $\mathcal{L}(D) = 0$ and $|f(\theta)| \leq c$ for $\theta \in [-\pi, \pi] - D$ (Halmos, 1958, p. 86). Obviously, if f is essentially bounded the set where f is not finite has null measure.

Remark 4. If $\mathbf{c} \in L_2^n([-\pi, \pi], \mathbb{C})$ and $|\mathbf{c}(\theta)|^2$ is essentially bounded, then, since the entries of Σ_n^x have finite integral, the variance of $\underline{\mathbf{c}}(L)\mathbf{x}_{nt}$ is finite. In particular,

$$\mathrm{var}(\underline{\mathbf{p}}_{nj}^x(L)\mathbf{x}_{nt}) = \int_{-\pi}^{\pi} \mathbf{p}_{nj}^x(\theta)\Sigma_n^x(\theta)\tilde{\mathbf{p}}_{nj}^x(\theta)d\theta = \int_{-\pi}^{\pi} \lambda_{nj}^x(\theta)d\theta < \infty.$$

We will invoke the following two results, the first known as the Lebesgue dominated convergence theorem.

Fact L_1. *Assume that $\{f_n,\ n \in \mathbb{N}\}$ is a sequence of integrable functions (i.e. having finite integral) defined on $[-\pi, \pi]$, such that (a) $\lim_n f_n(\theta) = f(\theta)$ a.e. in $[-\pi, \pi]$, and (b) $|f_n(\theta)| \leq g(\theta)$ a.e. in $[-\pi, \pi]$, where g is non-negative and integrable. Then f is integrable and*

$$\lim_n \int_{-\pi}^{\pi} f_n(\theta)d\theta = \int_{-\pi}^{\pi} f(\theta)d\theta$$

(see Apostol, 1974, p. 270).

Fact L_2. *Suppose that $\{f_n,\ n \in \mathbb{N}\}$ is a sequence of integrable functions defined on $[-\pi, \pi]$, that f is an integrable function defined on $[-\pi, \pi]$, and that*

$$\lim_n \int_{-\pi}^{\pi} |f_n(\theta) - f(\theta)|^2 d\theta = 0 \quad \text{or} \quad \lim_n \int_{-\pi}^{\pi} |f_n(\theta) - f(\theta)|d\theta = 0.$$

Then there exists an increasing sequence s_n such that $\lim_n f_{s_n}(\theta) = f(\theta)$ a.e. in $[-\pi, \pi]$ (see Apostol, 1974, p. 298; Halmos, 1958, p. 103, Theorem A, p. 93, Theorem D, p. 89, Theorem B).

Theorem 1. $\mathbf{x}$ *is idiosyncratic if and only if λ_1^x is essentially bounded.*
Proof. Let $\{\mathbf{a}_n(L),\ n \in \mathbb{N}\}$ be a DAS. By Fact M_1 (Appendix),

$$||\mathbf{a}(L)_n \mathbf{x}_{s_n t}||^2 = \int_{-\pi}^{\pi} \mathbf{a}_n^{\circ}(\theta)\Sigma_{s_n}^x(\theta)\tilde{\mathbf{a}}_n^{\circ}(\theta)d\theta \leq \int_{-\pi}^{\pi} \lambda_{s_n 1}^x(\theta)|\mathbf{a}_n^{\circ}(\theta)|^2 d\theta \leq \int_{-\pi}^{\pi} \lambda_1^x(\theta)|\mathbf{a}_n^{\circ}(\theta)|^2 d\theta.$$

$$(3)$$

If λ_1^x is essentially bounded, the RHS of (3) tends to zero, so that x is idiosyncratic.

Conversely, assume that λ_1^x is not essentially bounded. This means that for any positive integer n, there exists an integer s_n such that

$$\mu_n = \mathcal{L}(\{\theta : \lambda_{s_n 1}^x(\theta) \geq n\}) > 0.$$

Define h_n by

$$h_n(\theta) = \begin{cases} 1/\sqrt{\mu_n} & \text{if } \lambda_{s_n 1}^x(\theta) \geq n \\ 0 & \text{otherwise,} \end{cases}$$

and $H_n = \int_{-\pi}^{\pi} \lambda_{s_n 1}^x(\theta) h_n(\theta)^2 d\theta$, so that $H_n \geq n$. Then define $\mathbf{b}_n(L) = \underline{h}_n(L)\underline{\mathbf{p}}_{s_n 1}^x(L)/\sqrt{H_n}$. Clearly $\mathbf{b}_n(L)$ is a DAS, while $||\mathbf{b}_n(L)\mathbf{x}_{s_n t}||^2 = 1$, so that x is not idiosyncratic.   **QED**

Corollary. *If* x *is idiosyncratic then*

$$\sup_n \int_{-\pi}^{\pi} \lambda_{n1}^x(\theta) d\theta = \lim_n \int_{-\pi}^{\pi} \lambda_{n1}^x(\theta) d\theta < \infty.$$

Proof. Since λ_1^x is essentially bounded, we have $\int_{-\pi}^{\pi} \lambda_1^x(\theta) d\theta < \infty$. λ_{n1}^x converges a.e. in $[-\pi, \pi]$ to λ_1^x and is bounded a.e. in $[-\pi, \pi]$ by λ_1^x. Applying Fact L_1,

$$\lim_n \int_{-\pi}^{\pi} \lambda_{n1}^x(\theta) d\theta = \int_{-\pi}^{\pi} \lambda_1^x(\theta) d\theta < \infty.$$

QED

The following example shows that the converse of the Corollary is false.

Example 2. Assume that x_{it} is orthogonal to x_{jt-k} for any k and any $i \neq j$, and suppose that the spectral density of the stationary process x_{it} is $|\theta|^\alpha$, for $|\theta| > 1/i$, zero otherwise, with $-1 < \alpha < 0$. In this case the matrix Σ_n^x is diagonal, $\lambda_1^x(\theta) = |\theta|^\alpha$ for $\theta \neq 0$, zero for $\theta = 0$, which is not essentially bounded. Thus x is not idiosyncratic, even though $\sup_n \int_{-\pi}^{\pi} \lambda_{n1}^x(\theta) d\theta = \int_{-\pi}^{\pi} |\theta|^\alpha d\theta < \infty$.

Definition 8. *Assume that* x *fulfills Assumption 1. Consider the orthogonal projection*

$$x_{it} = \text{proj}(x_{it}|\mathcal{G}(\mathbf{x})) + \delta_{it}. \tag{4}$$

Decomposition (4) will be called the **canonical decomposition** *of* x.

4. A Finite Number of Dynamic Common Factors
4.1 Let us now give a formal definition of the generalized dynamic factor model and now state our main results.

Definition 9. *Let* q *be a non-negative integer. The double sequence* x *is a* q-**dynamic factor sequence**, q-*DFS henceforth, if* $L_2(\mathcal{P}, \mathbb{C})$ *contains an orthonormal* q-*dimensional white-noise vector process*

$$\mathbf{u} = \{(\, u_{1t} \quad u_{2t} \quad \cdots \quad u_{qt} \,)', \ t \in \mathbb{Z}\} = \{\mathbf{u}_t, \ t \in \mathbb{Z}\},$$

and a double sequence $\boldsymbol{\xi} = \{\xi_{it}, \ i \in \mathbb{N}, \ t \in \mathbb{Z}\}$ *fulfilling Assumption 1, such that:*

(i) for any $i \in \mathbb{N}$,

$$x_{it} = \chi_{it} + \xi_{it}$$
$$\chi_{it} = b_{i1}(L)u_{1t} + b_{i2}(L)u_{2t} + \cdots + b_{iq}(L)u_{qt} = \mathbf{b}_i(L)\mathbf{u}_t, \tag{5}$$

where $\mathbf{b}_i^\circ \in L_2([-\pi,\ \pi], \mathbb{C}^q)$;
(ii) λ_1^ξ is essentially bounded, i.e. $\boldsymbol{\xi}$ is idiosyncratic;
(iii) putting $\boldsymbol{\chi} = \{\chi_{it},\ i \in \mathbb{N},\ t \in \mathbb{Z}\}$, $\lambda_q^\chi(\theta) = \infty$ a.e. in $[-\pi,\ \pi]$.
The double sequences $\boldsymbol{\chi}$ and $\boldsymbol{\xi}$ are referred to as the common and the idiosyncratic component of representation (5).

Theorem 2. *The double sequence $\mathbf{x}$ is a q-DFS if and only if:*
(I) λ_{q+1}^x is essentially bounded;
(II) $\lambda_q^x = \infty$ a.e. in $[-\pi,\ \pi]$.

Remark 5. Forni, Hallin, Lippi and Reichlin (1998) propose a heuristic criterion to determine in empirical cases the number q such that (I) and (II) hold. Since they only rely on the 'only if' part of Theorem 2, their criterion provides evidence on the number of common factors, *under the assumption of a generalized dynamic factor model.* Once the 'if' part is proved, evidence that for some q (I) and (II) hold becomes evidence both that the series follow a generalized dynamic factors, and that the number of factors is q.

Theorem 3. *If $\mathbf{x}$ is a q-DFS with representation (5) then*

$$\overline{span}(\boldsymbol{\chi}) = \overline{span}(\mathbf{u}) = \mathcal{G}(\mathbf{x}).$$

Moreover

$$\chi_{it} = \mathrm{proj}(x_{it}|\mathcal{G}(\mathbf{x})). \tag{6}$$

An immediate but very important consequence of (6) is that if $\mathbf{x}$ is a q-DFS then the components χ_{it} and ξ_{it} are uniquely determined. Precisely:

Theorem 4. *Suppose that $\mathbf{x}$ is a q-DFS with representation (5). Suppose further that there exists an s-dimensional orthonormal white-noise vector process $\mathbf{v}$, with $v_{jt} \in L_2(\mathcal{P}, \mathbb{C})$, such that*

$$x_{it} = \omega_{it} + \zeta_{it}$$
$$\omega_{it} = \mathbf{c}_i(L)\mathbf{v}_t,$$

where $\mathbf{c}_i^\circ \in L_2([-\pi,\ \pi], \mathbb{C}^s)$, and that λ_1^ζ and λ_s^ω fulfill, respectively, conditions (ii) and (iii) of Definition 9. Then $s = q$, $\omega_{it} = \chi_{it}$ and $\zeta_{it} = \xi_{it}$.

Remark 6. It must be pointed out that the components are identified, not $\mathbf{u}_t$ or the filters $\mathbf{b}_i(L)$. If (5) holds, and $\mathbf{C}(L)$ is $q \times q$ and such that $\mathbf{C}^\circ(\theta)\tilde{\mathbf{C}}^\circ(\theta) = \mathbf{I}_q$, then $\chi_{it} = \mathbf{b}_i^*(L)\mathbf{u}_t^*$, with $\mathbf{b}_i^*(L) = \mathbf{b}_i(L)\mathbf{C}(L)$ and $\mathbf{u}_t^* = \tilde{\mathbf{C}}(L^{-1})\mathbf{u}_t$.

Remark 7. Note that in Definition 9 the filters $b_{ij}(L)$ are in general bilateral. If representation (5) must have a structural interpretation then it is reasonable to assume that the filters $b_{ij}(L)$ are one-sided. However, one-sidedness of the $b_{ij}(L)$ has no consequences on the eigenvalues λ_{nj}, nor fulfillment of conditions (I) and (II) has implications on the existence of one-sided representations of the common component. In this paper we deal only with

the number of common shocks, i.e. the dimension of $\mathbf{u}_t$, which is identified (Theorems 2, 3, 4), and with the reconstruction of χ_{it} and ξ_{it} (Theorem 5). Existence and identification of one-sided representations of the common component are left to further study.

Remark 8. The result in Theorem 4 can be restated by saying that if $\mathbf{x}$ is q-DFS, then q is minimal, i.e. no representation fulfilling Definition 9 is possible with a smaller number of factors. It is important to point out that condition (iii) in Definition 9 is crucial. For example, suppose that
$$x_{it} = b_i u_t + \xi_{it},$$
with $\boldsymbol{\xi}$ idiosyncratic and $\sum |b_i|^2 < \infty$. In this case $\lambda_1^\chi < \infty$. As a consequence, $b_i u_t + \xi_{it}$ is idiosyncratic, so that a representation with zero factor is possible.

Remark 9. Suppose that $\mathbf{x}_{nt}$ is a vector white noise for any n, so that the model is "isomorphic" to the static model in Chamberlain and Rothschild (1983). Then the eigenvalues λ_{nj}^x are constant as functions of θ. As a consequence, if $\lambda_s^x < \infty$, the model has q factors, with $q < s$. Unfortunately, in the general dynamic case, there exist cases where λ_s^x is essentially bounded, but the sequence does not fulfill Definition 9 for any $q < s$. Consider

$$x_{it} = b(L)u_t + \xi_{it},$$

with $\boldsymbol{\xi}$ idiosyncratic and
$$b^\circ(\theta) = \begin{cases} 1 \text{ if } \theta \in [-1,1] \\ 0 \text{ otherwise.} \end{cases}$$

In this case $\lambda_2^x(\theta)$ is essentially bounded, but $\lambda_1^x(\theta)$ is infinite for $\theta \in [-1,1]$, finite elsewhere. However, such cases do not seem to deserve further consideration.

The proof of Theorems 2 and 3 will require several steps. In Section 4.2 we introduce an additional assumption on $\mathbf{x}$ and show that it does not imply any loss of generality. In Section 4.3 we prove that conditions (I) and (II) are necessary for a q-DFS, which is relatively easy. The converse is much more complicated. In 4.4 we prove that $\mathcal{G}(\mathbf{x})$ contains a q-dimensional orthonormal white-noise vector process $\mathbf{z}$, so that $\mathcal{G}(\mathbf{x}) \supseteq \overline{\text{span}}(\mathbf{z})$. In 4.5 we prove that actually $\mathcal{G}(\mathbf{x}) = \overline{\text{span}}(\mathbf{z})$, so that the canonical decomposition has the form

$$x_{it} = \text{proj}(x_{it}|\mathcal{G}(\mathbf{x})) + \delta_{it} = \mathbf{c}_i(L)\mathbf{z}_t + \delta_{it}.$$

Lastly, in 4.6 we show that $\boldsymbol{\delta}$ is idiosyncratic, thus completing the proof of Theorem 2. In 4.7 we prove Theorem 3.

4.2 Theorems 2 and 3 will be proved supposing that

Assumption 2. *For any $n \in \mathbb{N}$, $j \leq n$ and $\theta \in [-\pi, \pi]$, $\lambda_{nj}^x(\theta) \geq 1$.*

To show that Assumption 2 does not imply any loss of generality, observe that, possibly by embedding $\mathcal{P}$ into a larger probability space, we can assume that $L_2(\mathcal{P}, \mathbb{C})$ contains a stationary sequence $\{\hat{\xi}_{it}, \ i \in \mathbb{N}, \ t \in \mathbb{Z}\}$ such that $\hat{\xi}_{it} \perp \mathbf{X}$ for any i and t, $\text{var}(\hat{\xi}_{it}) = 1$ for any i and t, and $\hat{\xi}_{it} \perp \hat{\xi}_{jt-k}$ for any t and $i \neq j$. Now define $\mathbf{y} = \{x_{it} + \hat{\xi}_{it}, \ i \in \mathbb{N}, \ t \in \mathbb{Z}\}$, and suppose that Theorems 2 and 3 have been proved under Assumption 2. We have:

(a) $\Sigma_n^y = \Sigma_n^x + I_n$, $\lambda_{nj}^y = \lambda_{nj}^x + 1$. Thus if conditions (I) and (II) hold for $\mathbf{x}$, then they hold for $\mathbf{y}$ as well. By Theorem 2 $\mathbf{y}$ is a q-DFS with representation $y_{it} = \check{\chi}_{it} + \check{\xi}_{it}$. By Theorem

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3, $\check{\chi}_{it} = \text{proj}(y_{it}|\mathcal{G}(\mathbf{y}))$. But the definitions of $\hat{\xi}$ and $\mathbf{y}$ imply that $\check{\chi}_{it} = \text{proj}(x_{it}|\mathcal{G}(\mathbf{x}))$. Therefore

$$x_{it} = \text{proj}(x_{it}|\mathcal{G}(\mathbf{x})) + (\check{\xi}_{it} - \hat{\xi}_{it}). \tag{7}$$

Since $\hat{\xi}_{it}$ is orthogonal to $\mathbf{X}$ and $\check{\xi} - \hat{\xi}$ is idiosyncratic, then (7) is a q-DFS representation. Thus if (I) and (II) hold for $\mathbf{x}$, then $\mathbf{x}$ has a q-DFS representation.

(b) If $\mathbf{x}$ has the q-DFS representation $x_{it} = \chi_{it} + \xi_{it}$, then $\mathbf{y}$ has the q-DFS representation $y_{it} = \chi_{it} + (\xi_{it} + \hat{\xi}_{it})$. Applying Theorem 2 to $\mathbf{y}$, we obtain conditions (I) and (II) for $\lambda_q^y = \lambda_q^x + 1$ and $\lambda_{q+1}^y = \lambda_{q+1}^x + 1$ and therefore for λ_q^x and λ_{q+1}^x. In conclusion, if Theorems 2 and 3 hold under Assumption 2, then Theorem 2 holds in general.

(c) In the same way, applying Theorems 2 and 3, supposedly proved under Assumption 2, to $\mathbf{y}$, Theorem 3 can be proved in general.

4.3 Let us prove that if $\mathbf{x}$ is a q-DFS then (I) and (II) hold. By Definition 9, $\Sigma_n^x(\theta) = \Sigma_n^\chi(\theta) + \Sigma_n^\xi(\theta)$. By Fact M$_3$ (Appendix), $\lambda_{nq}^x(\theta) \geq \lambda_{nq}^\chi(\theta)$, so that (II) is proved. Moreover, by Fact M$_3$,

$$\lambda_{nq+1}^x(\theta) \leq \lambda_{nq+1}^\chi(\theta) + \lambda_{n1}^\xi(\theta) = \lambda_{n1}^\xi(\theta), \tag{8}$$

so that (I) is proved. Moreover, (8) implies the following interesting inequality:

$$\lambda_{q+1}^x(\theta) \leq \lambda_1^\xi(\theta) \tag{9}$$

(the opposite inequality is proved in 4.7).

4.4 Now we start assuming (I) and (II). Firstly we prove that $\mathcal{G}(\mathbf{x})$ contains a q-dimensional white-noise vector. The proof goes as follows. We start with a q-dimensional orthonormal white noise, call it ψ_t, whose entries are linear combinations of the principal components of $\mathbf{x}_{mt}$, i.e. $\underline{\mathbf{p}}_{mj}^x(L)\mathbf{x}_{mt}$, for $j = 1, \ldots, q$, $t \in \mathbb{Z}$. Then we project ψ_t on the space spanned by the principal components of $\mathbf{x}_{nt}$, i.e. $\underline{\mathbf{p}}_{nj}^x(L)\mathbf{x}_{nt}$, $j = 1, \ldots, q$, $t \in \mathbb{Z}$, for $n > m$, call $\mathbf{y}_t$ the projection. We show that when m and n become large the distance between ψ_t and $\mathbf{y}_t$ becomes small. This leads to the construction of a sequence of q-dimensional white noise vectors whose components are Cauchy sequences and converge to $\mathcal{G}(\mathbf{x})$.

The proofs would be considerably easier if we could assume that $\lambda_{nq}^x(\theta) \geq \alpha_n$ a.e. in $[-\pi, \pi]$, where $\lim_n \alpha_n = \infty$. However, this condition is false in this 1-factor model:

$$x_{it} = (1 - L)u_t + \xi_{it}, \tag{10}$$

with $\Sigma_n^\xi = \mathbf{I}_n$, in which Σ_n^x is continuous and $\Sigma_n^x(0) = \mathbf{I}_n$ for any n. Unfortunately, to include cases like (10) our proofs must be carried over piecewise on $[-\pi, \pi]$.

We need some further notation and definitions. For $n \geq q$, we denote by $\mathbf{P}_n$ the $q \times n$ matrix

$$(\mathbf{p}_{n1}^{x}{}' \quad \mathbf{p}_{n2}^{x}{}' \quad \cdots \quad \mathbf{p}_{nq}^{x}{}')',$$

i.e. the matrix having the dynamic eigenvectors $\mathbf{p}_{nj}^x$, $j = 1, \ldots, q$, on the rows, and by $\mathbf{Q}_n$ the $(n - q) \times n$ matrix

$$(\mathbf{p}_{nq+1}^{x}{}' \quad \mathbf{p}_{nq+2}^{x}{}' \quad \cdots \quad \mathbf{p}_{nn}^{x}{}')'.$$

Moreover, let us call $\mathbf{\Lambda}_n$ the $q \times q$ diagonal matrix having on the diagonal the eigenvalues λ_{nj}^x, $j = 1, \ldots, q$ and by $\mathbf{\Phi}_n$ the $(n - q) \times (n - q)$ diagonal matrix having on the diagonal the

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eigenvalues λ_{nj}^x, $j = q+1, \ldots, n$. The matrices $\boldsymbol{\Sigma}_n^x$ and $\mathbf{I}_n$ can be rewritten in their spectral decomposition form:

$$\boldsymbol{\Sigma}_n^x = \tilde{\mathbf{P}}_n \boldsymbol{\Lambda}_n \mathbf{P}_n + \tilde{\mathbf{Q}}_n \boldsymbol{\Phi}_n \mathbf{Q}_n$$
$$\mathbf{I}_n = \tilde{\mathbf{P}}_n \mathbf{P}_n + \tilde{\mathbf{Q}}_n \mathbf{Q}_n. \tag{11}$$

Since $\boldsymbol{\Lambda}_n^{-1}(\theta)$ is bounded in $[-\pi, \pi]$ by Assumption 2, the definition

$$\boldsymbol{\psi}_t^n = (\,\psi_{1t}^n \quad \cdots \quad \psi_{qt}^n\,)' = \underline{\boldsymbol{\Lambda}}_n^{-1/2}(L)\underline{\mathbf{P}}_n(L)\mathbf{x}_{nt}$$

makes sense and $\boldsymbol{\psi}_t^n$ is an orthonormal white noise. Note that the processes ψ_{jt}^n, $j = 1, \ldots, q$, are the first q dynamic principal components of $\mathbf{x}_{nt}$, rescaled to get unit spectral density at any frequency.

In the sequel it will be convenient to write matrix products $\mathbf{AB}$ in which the number of columns of $\mathbf{A}$ is smaller than the number of rows of $\mathbf{B}$. In this case we implicitly assume that $\mathbf{A}$ has been augmented with columns of zeros to match the number of rows of $\mathbf{B}$. For example, we write $\underline{\mathbf{P}}_m(L)\mathbf{x}_{nt}$ for $n > m$, this meaning nothing other than $\underline{\mathbf{P}}_m(L)\mathbf{x}_{mt}$. In the same way, we have equations with a $1 \times m$ matrix on one side and a $1 \times n$ matrix on the other, with $m < n$, this meaning that the $1 \times m$ matrix has been augmented with zeros.

Now let $\mathbf{C}$ be a $q \times q$ matrix such that the entries c_{ij} belong to $L_2([-\pi, \pi], \mathbb{C})$ and are essentially bounded in modulus, so that the linear combination $\underline{\mathbf{C}}(L)\boldsymbol{\psi}_t^m$ has finite variance-covariance matrix (see Remark 4). We want to determine the (element by element) orthogonal projection of the vector $\underline{\mathbf{C}}(L)\boldsymbol{\psi}_t^m$ on the space

$$\overline{\mathrm{span}}(\{\psi_{jt}^n, \ j = 1, \ldots, q, \ t \in \mathbb{Z}\})$$

for $n > m$. From (11) we get

$$\mathbf{x}_{nt} = \tilde{\underline{\mathbf{P}}}_n(L)\underline{\mathbf{P}}_n(L)\mathbf{x}_{nt} + \tilde{\underline{\mathbf{Q}}}_n(L)\underline{\mathbf{Q}}_n(L)\mathbf{x}_{nt} = \tilde{\underline{\mathbf{P}}}_n(L)\underline{\boldsymbol{\Lambda}}_n^{1/2}(L)\boldsymbol{\psi}_t^n + \tilde{\underline{\mathbf{Q}}}_n(L)\underline{\mathbf{Q}}_n(L)\mathbf{x}_{nt}. \tag{12}$$

Since $\mathbf{Q}_n(\theta)\boldsymbol{\Sigma}_n^x(\theta)\tilde{\mathbf{P}}_n(\theta) = \boldsymbol{\Phi}_n(\theta)\mathbf{Q}_n(\theta)\tilde{\mathbf{P}}_n(\theta) = 0$ for any θ, the two terms on the RHS of (12) are orthogonal at any lead and lag element by element, so that the first is the projection of $\mathbf{x}_{nt}$ on $\overline{\mathrm{span}}(\{\psi_{jt}^n, \ j = 1, \ldots, q, \ t \in \mathbb{Z}\})$ and the second is the residual. The required projection equation is then obtained by applying on both sides the operator $\underline{\mathbf{C}}(L)\underline{\boldsymbol{\Lambda}}_m^{-1/2}(L)\underline{\mathbf{P}}_m(L)$ and noting that $\underline{\boldsymbol{\Lambda}}_m^{-1/2}(L)\underline{\mathbf{P}}_m(L)\mathbf{x}_{nt} = \underline{\boldsymbol{\Lambda}}_m^{-1/2}(L)\underline{\mathbf{P}}_m(L)\mathbf{x}_{mt} = \boldsymbol{\psi}_t^m$, i.e.

$$\underline{\mathbf{C}}(L)\boldsymbol{\psi}_t^m = \underline{\mathbf{D}}(L)\boldsymbol{\psi}_t^n + \underline{\mathbf{R}}(L)\mathbf{x}_{nt}, \tag{13}$$

where

$$\mathbf{D} = \mathbf{C}\boldsymbol{\Lambda}_m^{-1/2}\mathbf{P}_m\tilde{\mathbf{P}}_n\boldsymbol{\Lambda}_n^{1/2}, \quad \mathbf{R} = \mathbf{C}\boldsymbol{\Lambda}_m^{-1/2}\mathbf{P}_m\tilde{\mathbf{Q}}_n\mathbf{Q}_n. \tag{14}$$

Note that $\mathbf{D}$, as well as $\boldsymbol{\Delta}$, $\mathbf{H}$ and $\mathbf{F}$, which are defined below, depend on $\mathbf{C}$, m and n. However, as no confusion can arise, we do not explicit this dependence for notational simplicity. The following result holds.

Lemma 5. *Suppose that (I) and (II) hold. Assume $n > m$ and let $\mathbf{C}(\theta)$ be a unitary matrix, i.e. $\mathbf{C}(\theta)\tilde{\mathbf{C}}(\theta) = \mathbf{I}_q$ for any $\theta \in [-\pi, \pi]$. Consider the projection equation*

$$\underline{\mathbf{C}}(L)\boldsymbol{\psi}_t^m = \underline{\mathbf{D}}(L)\boldsymbol{\psi}_t^n + \underline{\mathbf{R}}(L)\mathbf{x}_{nt}, \tag{15}$$

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where $\mathbf{D}$ and $\mathbf{R}$ are defined as in (14), and call $\mu(\theta)$ the first eigenvalue of the spectral density matrix of the residual $\underline{\mathbf{R}}(L)\mathbf{x}_{nt}$. Then $\mu(\theta) \leq \lambda_{nq+1}^x(\theta)/\lambda_{mq}^x(\theta)$.

Proof. The matrix $\mathbf{I}_n - \tilde{\mathbf{Q}}_n \mathbf{Q}_n$ is non-negative definite by (11) and $\lambda_{nq+1}^x \tilde{\mathbf{Q}}_n \mathbf{Q}_n - \tilde{\mathbf{Q}}_n \mathbf{\Phi}_n \mathbf{Q}_n$ is non-negative definite by the definition of $\mathbf{\Phi}_n$, so that $\lambda_{nq+1}^x \mathbf{I}_n - \tilde{\mathbf{Q}}_n \mathbf{\Phi}_n \mathbf{Q}_n$ is also non-negative definite. Premultiplying by $\mathbf{C}\mathbf{\Lambda}_m^{-1/2}\mathbf{P}_m$ and postmultiplying by $\tilde{\mathbf{P}}_m \mathbf{\Lambda}_m^{-1/2} \tilde{\mathbf{C}}$ it is seen that

$$\lambda_{nq+1}^x \mathbf{C}\mathbf{\Lambda}_m^{-1}\tilde{\mathbf{C}} - \mathbf{R}\mathbf{\Sigma}_n^x\tilde{\mathbf{R}}$$

is also non-negative definite. The desired inequality follows from Fact M_3 (third and fourth inequality). **QED**

Now let us begin the construction of our converging sequence. Note that, under assumptions (I) and (II), there exists a set $\Pi \subseteq [-\pi, \pi]$ and a real W such that $[-\pi, \pi] - \Pi$ has null measure and, for $\theta \in \Pi$: (1) $\lambda_{nq+1}^x(\theta) \leq W$ for any $n \in \mathbb{N}$ and any $\theta \in \Pi$; (2) $\lambda_q^x(\theta) = \infty$ for $\theta \in \Pi$. Obviously, if a statement holds a.e. in Π, then it holds a.e. in $[-\pi, \pi]$, and vice versa.

Let M be a positive measure subset of Π such that $\lambda_{nq}^x(\theta) \geq \alpha_n$ for $\theta \in M$, where $\{\alpha_n, \ n \in \mathbb{N}\}$ is a real positive non-decreasing sequence satisfying $\lim_n \alpha_n = \infty$. Lastly, denote by K_M the set including all of the $q \times q$ matrices $\mathbf{C}$ with elements in $L_2([-\pi, \pi], \mathbb{C})$ such that (i) $\mathbf{C}(\theta) = 0$ for $\theta \notin M$, (ii) $\mathbf{C}(\theta)$ is unitary, i.e. $\mathbf{C}(\theta)\tilde{\mathbf{C}}(\theta) = \mathbf{I}_q$, for $\theta \in M$.

Consider (13) and assume $\mathbf{C} \in K_M$. Taking the spectral density of both sides we get, for $\theta \in M$,

$$\mathbf{I}_q = \mathbf{D}\tilde{\mathbf{D}} + \mathbf{R}\mathbf{\Sigma}_n^x\tilde{\mathbf{R}}. \tag{16}$$

Applying Lemma 5 we obtain $\mu(\theta) \leq \lambda_{nq+1}^x(\theta)/\lambda_{mq}^x(\theta) < W/\alpha_m$ for $\theta \in M$. Hence by Fact M_3, calling $\Delta_j(\theta)$, $j = 1, \ldots, q$, the eigenvalues of $\mathbf{D}(\theta)\tilde{\mathbf{D}}(\theta)$ in descending order, we have

$$1 \geq \Delta_q(\theta) > 1 - W/\alpha_m \tag{17}$$

for any θ in M. Thus, if m^* is such that

$$W/\alpha_{m^*} < 1,$$

we have

$$\Delta_q(\theta) > 1 - W/\alpha_{m^*} > 0 \tag{18}$$

everywhere in M for any $m \geq m^*$.

Now assume $m > m^*$. Denote by $\mathbf{\Delta}$ the diagonal matrix having Δ_j in place (j, j) and by $\mathbf{H}(\theta)$ a matrix which is measurable in M and fulfills for any $\theta \in M$: (a) $\mathbf{H}(\theta)\tilde{\mathbf{H}}(\theta) = \mathbf{I}_q$, (b) $\mathbf{H}(\theta)\mathbf{\Delta}(\theta)\tilde{\mathbf{H}}(\theta) = \mathbf{D}(\theta)\tilde{\mathbf{D}}(\theta)$. Inequality (18) ensures that $1/\sqrt{\Delta_j(\theta)}$ is bounded in M for $j = 1, \ldots, q$, so that the definition

$$\mathbf{F}(\theta) = \begin{cases} \mathbf{H}(\theta)\mathbf{\Delta}(\theta)^{-1/2}\tilde{\mathbf{H}}(\theta)\mathbf{D}(\theta) & \text{if } \theta \in M \\ 0 & \text{if } \theta \notin M \end{cases} \tag{19}$$

makes sense. Note that $\mathbf{F}$ belongs to K_M.

Lemma 6. *Suppose that (I) and (II) hold. Let M be a positive measure subset of Π and $\{\alpha_n, \ n \in \mathbb{N}\}$ a real positive non-decreasing sequence such that $\lim_n \alpha_n = \infty$. Assume that*

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(a) $\mathbf{C} \in K_M$;

(b) $\lambda_{nq}^x(\theta) \geq \alpha_n$ for $\theta \in M$;

Then, given τ, such that $2 > \tau > 0$, there exists an integer m_τ such that, firstly, $W/m_\tau < 1$, and, secondly, for $n > m \geq m_\tau$, the first eigenvalue of the spectral density matrix of

$$\underline{\mathbf{C}}(L)\psi_t^m - \underline{\mathbf{F}}(L)\psi_t^n$$

is less than τ for any $\theta \in \Pi$, where $\mathbf{F}$ is defined as in (19), with $\mathbf{D}$ defined as in (14).

Proof. From (13) we get

$$\underline{\mathbf{C}}(L)\psi_t^m - \underline{\mathbf{F}}(L)\psi_t^n = \underline{\mathbf{R}}(L)\mathbf{x}_{nt} + (\underline{\mathbf{D}}(L) - \underline{\mathbf{F}}(L))\psi_t^n.$$

The terms on the RHS are orthogonal at any lead and lag, so that the spectral density matrix of the sum is equal to the sum of the spectral density matrices. Hence, calling $\mathbf{S}$ the spectral density matrix on the LHS and using (16), we see that, for $\theta \in M$,

$$\mathbf{S} = 2\mathbf{I}_q - \mathbf{D}\tilde{\mathbf{F}} - \mathbf{F}\tilde{\mathbf{D}} = 2\mathbf{I}_q - 2\mathbf{H}\boldsymbol{\Delta}^{1/2}\tilde{\mathbf{H}} = 2\mathbf{H}(\mathbf{I}_q - \boldsymbol{\Delta}^{1/2})\tilde{\mathbf{H}},$$

whose larger eigenvalue is $2 - 2\sqrt{\Delta_q(\theta)}$, which is less than $2W/\alpha_m$ by (17). Thus, in order for $\mathbf{F}$ to make sense and the statement of the lemma to hold we need $2W/\alpha_{m_\tau} < \min(2, \tau)$. Since $\tau < 2$, m_τ must fulfill

$$2W/\alpha_{m_\tau} < \tau \tag{20}$$

QED

We need the following result, whose proof is given in the Appendix. Given the costationary processes $A = \{A_t, \ t \in \mathbb{Z}\}$, and $B = \{B_t, \ t \in \mathbb{Z}\}$ we denote by $\mathcal{S}(A, B; \theta)$ the value at the frequency θ of the cross spectrum between A and B.

Lemma 7. If $A = \{A_{nt}, \ t \in \mathbb{Z}\}$ and $B = \{B_{nt}, \ t \in \mathbb{Z}\}$ are costationary for $n \in \mathbb{N}$, $\lim_n A_{nt} = A_t$ and $\lim_n B_{nt} = B_t$ (in variance), then, for a sequence of integers s_i,

$$\lim_i \mathcal{S}(A_{s_i}, B_{s_i}; \theta) = \mathcal{S}(A, B; \theta),$$

a.e. in $[-\pi, \pi]$.

Lemma 8. Suppose that (I) and (II) hold and let M and $\{\alpha_n, \ n \in \mathbb{N}\}$ be as in Lemma 6. There exists a q-dimensional vector process $\mathbf{v}$ such that

(a) v_{jt} is an aggregate for $j = 1, \ldots, q$;

(b) the spectral density matrix of $\mathbf{v}$ equals $\mathbf{I}_q$ for θ a.e. in M, zero for $\theta \notin M$.

Proof. Let $\mathbf{F}_1$ be any element of K_M. Set $\tau = 1/2^3\pi$ and $s_1 = m_\tau$, where m_τ satisfies (20). Then set $\mathbf{G}_1(L) = \underline{\mathbf{F}}_1(L)\underline{\boldsymbol{\Lambda}}_{s_1}^{-1/2}\underline{\mathbf{P}}_{s_1}(L)$ and $\mathbf{v}_t^1 = \mathbf{G}_1(L)\mathbf{x}_{nt}$. It is easily seen that the spectral density matrix of $\mathbf{v}_t^1$ equals $\mathbf{I}_q$ for $\theta \in M$, zero for $\theta \notin M$.

Now set $\tau = 1/2^5\pi$ and $s_2 = m_\tau$, where m_τ satisfies (20). Then determine $\mathbf{D}$ as in (14), with $\mathbf{F}_1$ in place of $\mathbf{C}$, s_2 in place of n and $\mathbf{F}_2$ as in (19). Finally set $\mathbf{G}_2(L) = \underline{\mathbf{F}}_2(L)\underline{\boldsymbol{\Lambda}}_{s_2}^{-1/2}\underline{\mathbf{P}}_{s_2}(L)$ and $\mathbf{v}_t^2 = \mathbf{G}_2(L)\mathbf{x}_{nt}$. The spectral density matrix of $\mathbf{v}_t^2$ equals $\mathbf{I}_q$ for $\theta \in M$, zero for $\theta \notin M$. Moreover, by the definition of s_1 and Lemma 6, calling $\mathcal{A}_1$ the first eigenvalue of the spectral density matrix of $\mathbf{v}_t^1 - \mathbf{v}_t^2$, we have $\mathcal{A}_1(\theta) < 1/2^3\pi$ for any $\theta \in \Pi$, so that $\|v_{jt}^1 - v_{jt}^2\| < 1/2$, for $j = 1, \ldots, q$.

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By recursion, set $\tau = 1/2^{2k+1}\pi$ and $s_k = m_\tau$, where m_τ satisfies (20). Then determine $\mathbf{D}$ as in (14), with $\mathbf{F}_{k-1}$ in place of $\mathbf{C}$ and s_k in place of n, and $\mathbf{F}_k$ as in (19). Finally set $\mathbf{G}_k(L) = \underline{\mathbf{F}}_k(L)\mathbf{\Lambda}_{s_k}^{-1/2}\underline{\mathbf{P}}_{s_k}(L)$ and $\mathbf{v}_t^k = \mathbf{G}_k(L)\mathbf{x}_{nt}$. The spectral density matrix of $\mathbf{v}_t^k$ equals $\mathbf{I}_q$ for $\theta \in M$, zero for $\theta \notin M$. Moreover, by the definition of s_{k-1} and Lemma 6, calling $\mathcal{A}_{k-1}$ the first eigenvalue of the spectral density matrix of $\mathbf{v}_t^{k-1} - \mathbf{v}_t^k$, we have $\mathcal{A}_{k-1}(\theta) < 1/2^{2k-1}\pi$ for any $\theta \in \Pi$, so that $||v_{jt}^{k-1} - v_{jt}^k|| < 1/2^{k-1}$ for $j = 1, \ldots, q$.

Since we have

$$||v_{jt}^k - v_{jt}^{k+h}|| \leq ||v_{jt}^k - v_{jt}^{k+1}|| + \cdots + ||v_{jt}^{k+h-1} - v_{jt}^{k+h}|| < 1/2^{k-1},$$

then each component of $\{\mathbf{v}_t^k, \ k \in \mathbb{N}\}$ is a Cauchy sequence. Call $\mathbf{v}_t$ the vector of the limits. To prove (a), we have to show that each row of $\{\mathbf{G}_n(L), \ n \in \mathbb{N}\}$ is a DAS. We have

$$|\mathbf{G}_n^\circ(\theta)|^2 = \mathbf{F}_n(\theta)\mathbf{\Lambda}_n^{-1}(\theta)\tilde{\mathbf{F}}_n(\theta),$$

whose diagonal entries cannot be larger than $1/\lambda_{s_n q}^x(\theta)$ by Fact M_1, since $\mathbf{F}_n(\theta) \in K_M$. The latter ratio converges to zero a.e. in $[-\pi, \pi]$ and is less than 1 by Assumption 2, so that its integral on $[-\pi, \pi]$ converges to zero by Fact L_1.

Finally, (b) follows from Lemma 7 and the fact that the spectral density matrix of $\mathbf{v}_t^k$ equals $\mathbf{I}_q$ for $\theta \in M$, zero for $\theta \notin M$. **QED**

Lemma 9. *Suppose that (I) and (II) hold. There exists a q-dimensional orthonormal white-noise vector process $\mathbf{z}$ such that z_{jt} is an aggregate for $j = 1, \ldots, q$.*
Proof. Define ν_1 as the smallest among the integers m such that the measure of

$$\mathcal{L}(\{\theta \in \Pi, \ \lambda_{mq}^x(\theta) > 1\}) \geq \pi$$

and

$$M_1 = \{\theta \in \Pi, \ \lambda_{\nu_1 q}^x(\theta) > 1\}.$$

By recursion define ν_a, $a \in \mathbb{N}$, as the smallest among the integers m such that

$$\mathcal{L}(\{\theta \in M_{a-1}, \ \lambda_{mq}^x(\theta) > a\}) \geq \pi$$

and

$$M_a = \{\theta \in M_{a-1}, \ \lambda_{\nu_a q}^x(\theta) > a\}.$$

The measure of the set

$$N_1 = M_1 \cap M_2 \cap \cdots \cap M_a \cap \cdots$$

is not less than π. Now define N_2 starting with $\Pi - N_1$ and using $\mathcal{L}(\Pi - N_1)/2$ instead of π, N_a starting with $\Pi - N_1 - N_2 - \cdots - N_{a-1}$ and using $\mathcal{L}(\Pi - N_1 - N_2 - \cdots - N_{a-1})/2$. We have

$$\Pi = N_1 \cup N_2 \cup \cdots \cup N_a \cup \cdots$$

and

$$2\pi = \mathcal{L}(N_1) + \mathcal{L}(N_2) + \cdots + \mathcal{L}(N_a) + \cdots.$$

Lemma 8 can be applied to the subset N_a, with the sequence α_n suitably defined. We obtain a q-dimensional vector $\mathbf{v}_t^a = (v_{1t}^a \quad v_{2t}^a \quad \cdots \quad v_{qt}^a)'$ such that (i) v_{jt}^a is an aggregate for $j = 1, \ldots, q$; (ii) its spectral density matrix is $\mathbf{I}_q$ a.e. in N_a, zero for $\theta \notin N_a$. Now set

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$\mathbf{z}_t = \sum_{a=1}^{\infty} \mathbf{v}_t^a$. It is easily seen that the spectral density matrix of $\mathbf{z}_t$ is $\mathbf{I}_q$ a.e. in $[-\pi, \pi]$, so that $\mathbf{z}$ is a q-dimensional orthonormal white noise process. **QED**

4.5 We now prove that the space spanned by $\mathbf{z}$ is $\mathcal{G}(\mathbf{x})$.

Let y_t be an aggregate. Consider the projection

$$y_t = \text{proj}(y_t|\overline{\text{span}}(\mathbf{z})) + r_t.$$

We want to show that r_t is necessarily zero. Consider the $(q+1)$-dimensional vector process $(\,\mathbf{z}_t \quad r_t\,)$. Its spectral density, call it $\mathcal{W}$, is diagonal with $\mathbf{I}_q$ in the $q \times q$ upper-left submatrix, so that

$$\det \mathcal{W}(\theta) = \mathcal{S}(r_t, r_t; \theta).$$

Since z_{jt} and r_t belong to $\mathcal{G}(\mathbf{x})$, let $\{\mathbf{a}_{nj}(L), \ n \in \mathbb{N}\}$, for $j = 1, \ldots, q+1$ be DAS's such that

$$\lim_n \mathbf{a}_{nj}(L)\mathbf{x}_{s_n t} = z_{jt}, \text{ for } j = 1, \ldots, q,$$

$$\lim_n \mathbf{a}_{nq+1}(L)\mathbf{x}_{s_n t} = r_t.$$

Note that, possibly by augmenting the filters $\mathbf{a}_{nj}(L)$ with zeros, we can assume the same sequence $\{s_n, \ n \in \mathbb{N}\}$ for all $j = 1, \ldots, q+1$. Moreover: (1) $\int_{-\pi}^{\pi} |\mathbf{a}_{nj}^{\circ}(\theta)|^2 d\theta$ converges to zero for $j = 1, \ldots, q+1$, so that a subsequence of $\mathbf{a}_{nj}^{\circ}$ converges to zero a.e. in $[-\pi, \pi]$ (Fact L_2); (2) calling $\mathcal{Z}_n$ the spectral density matrix of the vector process

$$(\,\mathbf{a}_{n1}(L)\mathbf{x}_{s_n t} \quad \mathbf{a}_{n2}(L)\mathbf{x}_{s_n t} \quad \cdots \quad \mathbf{a}_{nq+1}(L)\mathbf{x}_{s_n t}\,),$$

a subsequence of $\mathcal{Z}_n$ converges to $\mathcal{W}$ a.e. in $[-\pi, \pi]$ (Lemma 7). Thus, with no loss of generality we can assume that $\mathbf{a}_{nj}^{\circ}$ converges to zero and $\mathcal{Z}_n$ converges to $\mathcal{W}$ a.e. in $[-\pi, \pi]$.

Now, for $j = 1, \ldots, q+1$, set $\mathbf{f}_{nj} = \mathbf{a}_{nj}^{\circ}\tilde{\mathbf{P}}_{s_n}$ and $\mathbf{g}_{nj} = \mathbf{a}_{nj}^{\circ} - \mathbf{f}_{nj}\mathbf{P}_{s_n}$, so that

$$\mathbf{a}_{nj}^{\circ} = \mathbf{f}_{nj}\mathbf{P}_{s_n} + \mathbf{g}_{nj}$$

and

$$|\mathbf{a}_{nj}^{\circ}(\theta)|^2 = |\mathbf{f}_{nj}(\theta)|^2 + |\mathbf{g}_{nj}(\theta)|^2.$$

Since $\mathbf{a}_{nj}^{\circ}$ converges to zero a.e. in $[-\pi, \pi]$, then $\mathbf{g}_{nj}$ converges to zero a.e. in $[-\pi, \pi]$. Moreover, the definition of $\mathbf{g}_{nj}$ and $\mathbf{f}_{nj}$ implies that

$$\mathbf{a}_{nj}(L)\mathbf{x}_{s_n t} = \underline{\mathbf{f}}_{nj}(L)\underline{\mathbf{P}}_{s_n}(L)\mathbf{x}_{s_n t} + \underline{\mathbf{g}}_{nj}(L)\mathbf{x}_{s_n t}$$

is the orthogonal projection of the LHS on the space spanned by $\mathbf{p}_{s_n k}^x(L)\mathbf{x}_{s_n t}$, for $k = 1, \ldots, q$ and $t \in \mathbb{Z}$. As a consequence, the spectral density matrix $\mathcal{Z}_n$ is equal to the spectral density matrix of

$$(\,\underline{\mathbf{f}}_{n1}(L)\underline{\mathbf{P}}_{s_n}(L)\mathbf{x}_{s_n t} \quad \underline{\mathbf{f}}_{n2}(L)\underline{\mathbf{P}}_{s_n}(L)\mathbf{x}_{s_n t} \quad \cdots \quad \underline{\mathbf{f}}_{nq+1}(L)\underline{\mathbf{P}}_{s_n}(L)\mathbf{x}_{s_n t}\,),$$

call it $\mathcal{Z}_n^1$, plus the spectral density matrix of

$$(\,\underline{\mathbf{g}}_{n1}(L)\mathbf{x}_{s_n t} \quad \underline{\mathbf{g}}_{n2}(L)\mathbf{x}_{s_n t} \quad \cdots \quad \underline{\mathbf{g}}_{nq+1}(L)\mathbf{x}_{s_n t}\,),$$

call it $\mathcal{Z}_n^2$: $\mathcal{Z}_n = \mathcal{Z}_n^1 + \mathcal{Z}_n^2$.

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Now observe firstly that $\mathcal{Z}_n^1$ is singular for any θ. Secondly, since $\mathbf{g}_{nj}(\theta)$ is orthogonal to $\mathbf{p}_{s_n k}^x(\theta)$, for $k = 1, \ldots, q$, by Fact M_1 (Appendix),

$$\mathbf{g}_{nj}(\theta) \Sigma_{s_n}^x(\theta) \tilde{\mathbf{g}}_{nj}(\theta) \leq \lambda_{s_n q + 1}^x |\mathbf{g}_{nj}(\theta)|^2.$$

Essential boundedness of λ_{q+1}^x along with convergence to zero a.e. of $\mathbf{g}_{nj}$ imply that $\mathcal{Z}_n^2$ converges to zero a.e. in $[-\pi, \pi]$. This implies that $\det \mathcal{Z}_n$ converges to zero a.e. in $[-\pi, \pi]$ and therefore that $\det \mathcal{W}(\theta) = \mathcal{S}(r_t, r_t; \theta) = 0$ a.e. in $[-\pi, \pi]$, so that $r_t = 0$.

4.6 So far we have proved that if (I) and (II) hold then

$$x_{it} = \gamma_{it} + \delta_{it}$$
$$\gamma_{it} = \text{proj}(x_{it} | \mathcal{G}(\mathbf{x})) = \mathbf{c}_i(L) \mathbf{z}_t,$$

where $\mathbf{z}$ is a q-dimensional orthonormal white noise, and $\mathbf{c}_i \in L_2^q([-\pi, \pi], \mathbb{C})$. Suppose that δ is idiosyncratic. By Fact M_3, $\lambda_{nq}^\gamma(\theta) \geq \lambda_{nq}^x(\theta) - \lambda_{n1}^\delta(\theta)$, so that $\lambda_q^\gamma(\theta) = \infty$ a.e. in $[-\pi, \pi]$. Thus, to complete the proof of Theorem 2 we must only show that δ is idiosyncratic.

We need some additional preliminary results. Suppose that $\mathbf{v} = \{\mathbf{v}_t, \ t \in \mathbb{Z}\}$ and $\mathbf{w} = \{\mathbf{w}_t, \ t \in \mathbb{Z}\}$ are orthonormal q-dimensional white-noise vectors whose components belong to $L_2(\mathcal{P}, \mathbb{C})$. Moreover, suppose that $\mathbf{v}$ and $\mathbf{w}$ are costationary, i.e. $\mathrm{E}(\mathbf{v}_t \tilde{\mathbf{w}}_{t-k})$ does not depend on t, so that we can use the notation $\Gamma_k^{vw} = \mathrm{E}(\mathbf{v}_t \tilde{\mathbf{w}}_{t-k})$. Setting

$$\mathbf{A}(L) = \sum_{k=-\infty}^{\infty} \Gamma_k^{vw} L^k,$$

it is easily seen that $\mathbf{A}(L)\mathbf{w}_t$ is the orthogonal projection of $\mathbf{v}_t$ on the process $\mathbf{w}$. Moreover, $\tilde{\mathbf{A}}(F)\mathbf{v}_t$, where $F = L^{-1}$, is the orthogonal projection of $\mathbf{w}_t$ on the process $\mathbf{v}$. Lastly, consider the matrix $\mathbf{A}(e^{-i\theta})$. Its (i, j) entry is the cross spectrum between v_{it} and w_{jt} and has therefore modulus bounded by $1/2\pi$ for any $\theta \in [-\pi, \pi]$.

Definition 10. *For* $n = 1, 2, \ldots, \infty$, *let* $\mathbf{v}_n = \{\mathbf{v}_{nt}, \ t \in \mathbb{Z}\}$ *be a sequence of q-dimensional orthonormal white-noise vectors. Assume that* $\mathbf{v}_n$ *and* $\mathbf{v}_m$ *are costationary for all n and m. Consider the orthogonal projection*

$$\mathbf{v}_{mt} = \mathbf{A}^{mn}(L)\mathbf{v}_{nt} + \rho_t^{mn}, \tag{21}$$

and let $\mathcal{D}^{mn}$ *be the spectral density of* ρ_t^{mn}. *The sequence* $\{\mathbf{v}_n, \ n \in \mathbb{N}\}$ **generates a Cauchy sequence of spaces** *if, given* $\epsilon > 0$, *for θ a.e. in* $[-\pi, \pi]$ *there exists an integer $m_\epsilon(\theta)$ such that for $m > m_\epsilon(\theta)$,* $\text{trace}(\mathcal{D}^{mn}(\theta)) < \epsilon$.

Remark 10. Note that, if $\mathbf{v}_{nt}$ converges, it generates a Cauchy sequence of spaces, whereas the converse does not necessarily hold. As we shall show, the normalized principal components ψ_t^n generate a Cauchy sequence of spaces. However, they do not converge in general: for example, take $q = 1$ and assume that ψ_t^n is a normalized principal component converging to ψ_t; then $(-1)^n \psi_t^n$ is also a normalized principal component which does not converge.

Lemma 10. *Assume that* $\{\mathbf{v}_n, \ n \in \mathbb{N}\}$ *generates a Cauchy sequence of spaces and let* $y = \{y_t, \ t \in \mathbb{Z}\}$, *with $y_t \in L_2(\mathcal{P}, \mathbb{C})$, be costationary with $\mathbf{v}_n \in L_2(\mathcal{P}, \mathbb{C})$ for any n. Consider the orthogonal projections of y_t on the process $\mathbf{v}_n$, i.e. $Y_{nt} = \text{proj}(y_t | \overline{\text{span}}(\mathbf{v}_n))$. Then Y_{nt} converges in mean square to an element Y_t in $L_2(\mathcal{P}, \mathbb{C})$.*

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Proof. We have

$$y_t = Y_{nt} + r_{nt} = \mathbf{b}_n(L)\mathbf{v}_{nt} + r_{nt}$$

$$y_t = Y_{mt} + r_{mt} = \mathbf{b}_m(L)\mathbf{v}_{mt} + r_{mt},$$

where $\mathbf{b}_n(L)$ and $\mathbf{b}_m(L)$ are square summable q-dimensional filters, so that

$$\mathbf{b}_n(L)\mathbf{v}_{nt} - \mathbf{b}_m(L)\mathbf{v}_{mt} = r_{mt} - r_{nt}.$$

The spectral density of the LHS is the cross spectrum between the LHS and the RHS. The latter, due to the definition of r_{mt} and r_{nt}, is the sum of the cross spectrum between r_{nt} and $\mathbf{b}_m(L)\mathbf{v}_{mt}$, call it $\mathcal{S}_1$, and the cross spectrum between r_{mt} and $\mathbf{b}_n(L)\mathbf{v}_{nt}$, call it $\mathcal{S}_2$. Using (21), $\mathcal{S}_1$ is the cross spectrum between r_{nt} and $\mathbf{b}_m(L)\mathbf{A}^{mn}(L)\mathbf{v}_{nt} + \mathbf{b}_m(L)\boldsymbol{\rho}_t^{mn}$, which reduces to the cross spectrum between r_{nt} and $\mathbf{b}_m(L)\boldsymbol{\rho}_t^{mn}$, call it $\mathcal{C}_{mn}$. Now observe that both the spectral density of r_{nt} and the entries of $\mathbf{b}_m(e^{-i\theta})$ are bounded in modulus by the spectral density of y_t. Thus, since $\{\mathbf{v}_n, \ n \in \mathbb{N}\}$ generates a Cauchy sequence of spaces, $\mathcal{C}_{mn}$ converges to zero a.e. in $[-\pi, \ \pi]$ as $m, n \to \infty$. The same argument holds for $\mathcal{S}_2$, so that the spectral density of $Y_{nt} - Y_{mt}$ converges to zero a.e. in $[-\pi, \ \pi]$ as $m, n \to \infty$. Since both the spectral densities of Y_{nt} and of Y_{mt} are dominated by the spectral density of y_t, by the Lebesgue dominated convergence theorem (Fact L_1), the integral of the spectral density of $Y_{nt} - Y_{mt}$ also converges to zero as $m, n \to \infty$, so that Y_{nt} is a Cauchy sequence. **QED**

Now let us go back to equation (12) and concentrate on a single line, i.e. the orthogonal decomposition obtained by projecting x_{it} on the normalized principal components ψ_{jt}^n, $j = 1, \dots, q$. Calling $\underline{\boldsymbol{\pi}}_{ni}(L)$ the i-th (q-dimensional) row of $\underline{\tilde{\mathbf{P}}}_n(L)$ and $\underline{\mathbf{q}}_{ni}(L)$ the i-th row of $\underline{\tilde{\mathbf{Q}}}_n(L)$, we get

$$x_{it} = \underline{\boldsymbol{\pi}}_{ni}(L)\underline{\boldsymbol{\Lambda}}_n^{1/2}(L)\psi_t^n + \underline{\mathbf{q}}_{ni}(L)\underline{\mathbf{Q}}_n(L)\mathbf{x}_{nt}.$$

Lemma 11. *The sequence $\{\psi^n, \ n \in \mathbb{N}\}$ generates a Cauchy sequence of spaces.*
Proof. For $n > m$ consider (15) for $\underline{\mathbf{C}}(L) = \mathbf{I}_q$:

$$\psi_t^m = \underline{\mathbf{D}}(L)\psi_t^n + \rho_t^{mn}. \tag{22}$$

Calling $\mathcal{D}^{mn}$ the spectral density of ρ_t^{mn}, convergence to zero of $\text{trace}(\mathcal{D}^{mn}(\theta))$ for θ a.e. in $[-\pi, \ \pi]$ and $n > m$ is a consequence of Lemma 5. On the other hand,

$$\psi_t^n = \underline{\tilde{\mathbf{D}}}(F)\psi_t^m + \rho_t^{nm}. \tag{23}$$

From (22) and (23) we get

$$\mathbf{I}_q = \mathbf{D}(e^{-i\theta})\tilde{\mathbf{D}}(e^{i\theta}) + \mathcal{D}^{mn}(\theta) = \tilde{\mathbf{D}}(e^{i\theta})\mathbf{D}(e^{-i\theta}) + \mathcal{D}^{nm}(\theta).$$

By taking the trace on both sides and noting that the trace of $\mathbf{D}(e^{i\theta})\tilde{\mathbf{D}}(e^{-i\theta})$ is equal to the trace of $\tilde{\mathbf{D}}(e^{-i\theta})\mathbf{D}(e^{i\theta})$ we get

$$\text{trace}\left(\mathcal{D}^{mn}(\theta)\right) = \text{trace}\left(\mathcal{D}^{nm}(\theta)\right).$$

Thus $\text{trace}(\mathcal{D}^{mn}(\theta))$ converges to zero for any diverging n and m. **QED**

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The following theorem, besides being useful to show that δ is idiosyncratic, is important *per se*, because of its implications for the estimation of common and idiosyncratic components (see Forni, Hallin, Lippi and Reichlin, 1998).

Theorem 5. *The sequence of projections* $\gamma_{it}^n = \underline{\pi}_{ni}(L)\underline{\Lambda}_n^{1/2}(L)\psi_t^n = \underline{\pi}_{ni}(L)\underline{P}_n(L)\mathbf{x}_{nt}$, $n \in \mathbb{N}$ *converges in mean square to* $\gamma_{it} = \mathrm{proj}(x_{it}|\mathcal{G}(\mathbf{x}))$, *for any* i.

Proof. By Lemmas 10 and 11 γ_{it}^n converges in mean square to an element γ_{it}^* in $\mathbf{X}$. Therefore the sequence of the residuals $\delta_{it}^n = x_{it} - \gamma_{it}^n$ also converges to an element δ_{it}^* in $\mathbf{X}$. By Lemma 7, δ_{it}^* must be orthogonal to γ_{it}^* at all leads and lags. Moreover, γ_{it}^* is an aggregate, since $\underline{\pi}_{ni}(L)\underline{P}_n(L)$ is a DAS. To see this, consider that the spectral density of γ_{it}^n, i.e. $\pi_{in}\Lambda_n\tilde{\pi}_{in}$, is not larger than $\pi_{ni}\tilde{\pi}_{ni}\lambda_{nq}^x$, and is bounded above by the spectral density of x_{it}, call it σ_i, implying $\pi_{ni}(\theta)\tilde{\pi}_{ni}(\theta) \leq \sigma_i(\theta)/\lambda_{nq}^x(\theta)$. The latter ratio converges to zero a.e. in $[-\pi, \pi]$ and is bounded above by $\sigma_i(\theta)$ by Assumption 2, so that Fact L_1 applies. Summing up, γ_{it}^* belongs to $\mathcal{G}(\mathbf{x})$ and is orthogonal to δ_{it}^*, so that $\gamma_{it}^* = \gamma_{it}$. **QED**

The following Lemma concludes the proof of Theorem 2.

Lemma 12. δ *is idiosyncratic.*
Proof. Let us fix m and denote by Σ_m^δ the spectral density matrix of the vector process $\delta_{mt} = (\ \delta_{1t} \quad \delta_{2t} \quad \cdots \quad \delta_{mt}\ )'$. We want to show that the first eigenvalue of such matrix, i.e. $\lambda_{m1}^\delta(\theta)$, cannot be larger than $\sup_n \lambda_{nq+1}^x(\theta) = \lambda_{q+1}^x(\theta)$ for any $\theta \in [-\pi, \pi]$. Let $\Sigma_m^{\delta^n}$, $n > m$, be the spectral density matrix of $\delta_{mt}^n = (\ \delta_{1t}^n \quad \delta_{2t}^n \quad \cdots \quad \delta_{mt}^n\ )'$ and $\lambda_{m1}^{\delta^n}$ be its first eigenvalue. By Theorem 5 δ_{it}^n converges to δ_{it} in mean square for $i = 1, \ldots, m$, so that, by Lemma 7, a subsequence of $\Sigma_m^{\delta^n}$ converges to $\Sigma_m^{\delta^n}$ a.e. in $[-\pi, \pi]$. Assuming that $\lim_n \Sigma_m^{\delta^n} = \Sigma_m^\delta$ a.e. in $[-\pi, \pi]$ avoids further complication in notation and does not imply any loss of generality. Continuity of the eigenvalues as functions of the matrix entries (Ahlfors, pp. 300-6; see also the proof of Lemma 1) implies that

$$\lim_n \lambda_{m1}^{\delta^n}(\theta) = \lambda_{m1}^\delta(\theta), \tag{24}$$

a.e. in $[-\pi, \pi]$. Moreover, note that $\Sigma_m^{\delta^n}$ is the $m \times m$ upper-left submatrix of $\Sigma_n^{\delta^n}$, so that, by Fact M_1,

$$\lambda_{m1}^{\delta^n}(\theta) \leq \lambda_{n1}^{\delta^n}(\theta) = \lambda_{nq+1}^x(\theta)$$

for any $n \geq m$ and any θ in $[-\pi, \pi]$. Hence by (24) $\lambda_{m1}^\delta(\theta) \leq \lambda_{q+1}^x(\theta)$. Since this is true for any m,

$$\lambda_1^\delta(\theta) \leq \lambda_{q+1}^x(\theta), \tag{25}$$

so that λ_1^δ is essentially bounded. The statement follows from Theorem 1. **QED**

4.7 Now we prove Theorem 3. Assume that $\mathbf{x}$ fulfills Definition 9, so that

$$x_{it} = \chi_{it} + \xi_{it}$$
$$\chi_{it} = \mathbf{b}_i(L)\mathbf{u}_t,$$

where $\mathbf{u}$ is q-dimensional. By Theorem 2, $\mathbf{x}$ has also the representation

$$x_{it} = \gamma_{it} + \delta_{it}$$
$$\gamma_{it} = \mathrm{proj}(x_{it}|\mathcal{G}(\mathbf{x})) = \mathbf{c}_i(L)\mathbf{z}_t,$$

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where $\mathbf{z}$ is q-dimensional and

$$\overline{\mathrm{span}}(\mathbf{z}) = \mathcal{G}(\mathbf{x}).$$

Since $\boldsymbol{\xi}$ is idiosyncratic

$$\mathcal{G}(\mathbf{x}) \subseteq \overline{\mathrm{span}}(\boldsymbol{\chi}),$$

and obviously

$$\overline{\mathrm{span}}(\boldsymbol{\chi}) \subseteq \overline{\mathrm{span}}(\mathbf{u}).$$

On the other hand, since both $\mathbf{u}$ and $\mathbf{z}$ are q-dimensional, then

$$\overline{\mathrm{span}}(\mathbf{z}) = \mathcal{G}_(\mathbf{x}) = \overline{\mathrm{span}}(\boldsymbol{\chi}) = \overline{\mathrm{span}}(\mathbf{u}). \tag{26}$$

Now, (26) implies that $\chi_{it} \in \mathcal{G}(\mathbf{x})$ and $\xi_{it} \perp \mathcal{G}(\mathbf{x})$, so that $\chi_{it} = \mathrm{proj}(x_{it}|\mathcal{G}(\mathbf{x}))$ and $\xi_{it} = \delta_{it}$.

Remark 11. Since we have proved that $\delta_{it} = \xi_{it}$, (9) and (25) imply that

$$\lambda_{q+1}^{x}(\theta) = \lambda_{1}^{\xi}(\theta)$$

a.e. in $[-\pi, \pi]$.

5. Non-stationary variables

The case of trend stationary or difference stationary variables can be easily accommodated in our model. Assuming that the nature of non-stationarity is correctly detected, then, in the first case, i.e. $x_{it} = T_t + z_{it}$, where T_t is a deterministic trend, our results should be applied to the stationary components z_{it}. In the second case, assume, for the sake of simplicity, that the variables x_{it} are I(1). Consider the differences $y_{it} = (1 - L)x_{it}$ and suppose that (I) and (II) hold for λ_{q+1}^{y} and λ_{q}^{y} respectively. Then we have the representation

$$(1 - L)x_{it} = \chi_{it} + \xi_{it}$$
$$\chi_{it} = \mathbf{b}_i(L)\mathbf{u}_t,$$

where $\mathbf{u}_t$ is q-dimensional and $\boldsymbol{\xi}$ is idiosyncratic. Now observe that the vectors $\boldsymbol{\chi}_{nt}$ and $\boldsymbol{\xi}_{nt}$ are identified, and so are the spectral density matrices Σ_n^{χ} and Σ_n^{ξ}. Therefore all the information necessary to determine whether the χ's, or the ξ's, are I(1) or I(0), and whether cointegration relationships hold among the χ's or the ξ's, can be recovered starting with the spectral density matrices of the x's.

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Appendix

Proof of Lemma 1. Since the leading coefficient of the characteristic polynomial of Σ_n^x never vanishes, the eigenvalues, as functions of the entries of Σ_n^x, are continuous. Precisely, let $\mathcal{M}$ be the range of the function $\phi : [-\pi, \ \pi] \mapsto \mathbb{C}^{n \times n}$ that associates $\Sigma_n^x(\theta)$ with θ. There exist n functions $\delta_j : \mathcal{M} \mapsto \mathbb{R}$, that are continuous in $\mathcal{M}$ and such that $\delta_j(\phi(\theta))$ is a root of the characteristic equation of $\Sigma_n^x(\theta)$ (see Ahlfors, 1987, pp. 300-6). Now, consider the function $\nu_1 : \mathcal{M} \mapsto \mathbb{R}$ defined as

$$\nu_1(\mu) = \max_{j=1,n} \delta_j(\mu).$$

Obviously $\lambda_{n1}^x(\theta) = \nu_1(\phi(\theta))$. By a standard argument ν_1 is continuous on $\mathcal{M}$. Measurability of ϕ (see Assumption 1) ensures measurability of λ_{n1}^x. Now let $k(1,\mu)$ be the integer such that $\delta_{k(1,\mu)}(\mu) = \nu_1(\mu)$. Define

$$\nu_2(\mu) = \max_{\substack{j=1,n \\ j \neq k(1,\mu)}} \delta_j(\mu).$$

Obviously $\lambda_{n2}^x(\theta) = \nu_2(\phi(\theta))$ and the argument used for the first eigenvalue applies. Iteratively we obtain that all the eigenvalues λ_{nj}^x are measurable. **QED**

Proof of Lemma 2. Let $\mathbf{y} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}'$ be an n-dimensional stochastic vector with variance-covariance matrix Σ. Let λ_k be the k-th eigenvalue of Σ, in descending order, and $\mathbf{p}_k$ an eigenvector of Σ associated with λ_k. We recall that:

Fact M_1. For $k = 1, \ldots, n$, λ_k is

$$\max_{\mathbf{b}} ||\mathbf{by}||^2 = \max_{\mathbf{b}} \mathbf{b}\Sigma\tilde{\mathbf{b}}$$

$$s.t. \ \mathbf{b} \in \mathbb{C}^n, \ |\mathbf{b}| = 1, \ \mathbf{b} \perp \mathbf{p}_j \ for \ j < k.$$

Moreover, if $\mathbf{b} = \mathbf{p}_k$, then the orthogonality condition is fulfilled and $||\mathbf{p}_k\mathbf{y}||^2 = \lambda_k$ (see ???).

Fact M_2. For $k = 1, \ldots, n$, let $\hat{\mathbf{c}}$ be any $(k-1)$-tuple $\{\mathbf{c}_j, \ j = 1, \ldots, k-1\}$, where $\mathbf{c}_j \in \mathbb{C}^n$. The eigenvalue λ_k is

$$\min_{\hat{\mathbf{c}}} \max_{\mathbf{b}} \mathbf{b}\Sigma\tilde{\mathbf{b}}$$

$$s.t. \ |\mathbf{b}| = 1, \ \mathbf{b} \perp \mathbf{c}_j, \ j = 1, \ldots, k-1$$

(see Brillinger, 1981, p. 84, Exercise 3.10.16).

Going back to the proof, using Fact M_1 the statement of Lemma 2 is trivial for $k = 1$. For $k > 1$ consider:

$$\min_{\hat{\mathbf{c}}} \max_{\mathbf{b}} \mathbf{b}\Sigma_{n+1}^x(\theta)\tilde{\mathbf{b}}$$

$$s.t. \ (I) \ |\mathbf{b}| = 1, \ \mathbf{b} \perp \mathbf{c}_j, \ j = 1, \ldots, k-1; \ (II) \ \text{the last component of } \mathbf{b} \text{ is zero.} \tag{27}$$

Trivially, the value of (27) cannot exceed $\lambda_{n+1k}^x(\theta)$, as obtained using Fact M_2. On the other hand, the constraint on the last component of $\mathbf{b}$ implies that if $\mathbf{b} \perp \mathbf{c}_j$ then $\mathbf{b}$ is also orthogonal to the $k-1$ vectors of $\mathbb{C}^n$ whose components are the first n components of $\mathbf{c}_j$, for $j = 1, \ldots, k-1$. Thus the value of (27) is $\lambda_{nk}^x(\theta)$. **QED**

The following is a useful consequence of Fact M_2.

Fact M₃. *Let D and E be $m \times m$ Hermitian non-negative definite, and $F = D + E$. Then*

$$\lambda_s^f \leq \lambda_s^d + \lambda_1^e$$
$$\lambda_s^f \leq \lambda_1^d + \lambda_s^e$$
$$\lambda_s^f \geq \lambda_s^d$$
$$\lambda_s^f \geq \lambda_s^e$$

for any $s = 1, \ldots, m$.

Proof. For the first inequality observe that, calling $\mathcal{V}_s$ the set of all vectors $\mathbf{b} \in \mathbb{C}^m$ such that $|\mathbf{b}| = 1$ and $\mathbf{b} \perp \mathbf{c}_j$, where $\mathbf{c}_j$, $j = 1, \ldots, s - 1$, is a given $(s - 1)$-tuple of vectors in $\mathbb{C}^m$,

$$\max_{\mathbf{b} \in \mathcal{V}_s} \mathbf{b}\Sigma^f \tilde{\mathbf{b}} \leq \max_{\mathbf{b} \in \mathcal{V}_s} \mathbf{b}\Sigma^d \tilde{\mathbf{b}} + \max_{\mathbf{b} \in \mathcal{V}_s} \mathbf{b}\Sigma^e \tilde{\mathbf{b}} \leq \max_{\mathbf{b} \in \mathcal{V}_s} \mathbf{b}\Sigma^d \tilde{\mathbf{b}} + \max_{\mathbf{b} \in \mathcal{V}_1} \mathbf{b}\Sigma^e \tilde{\mathbf{b}} \leq \max_{\mathbf{b} \in \mathcal{V}_s} \mathbf{b}\Sigma^d \tilde{\mathbf{b}} + \lambda_1^e.$$

Then take the minimum over all $(s - 1)$-tuples. The second inequality follows in the same way. The second and third are trivial. **QED**

Proof of Lemma 3. Let $\mathbf{S}^1(\theta) = \mathbf{I}_n \lambda_{n1}^x(\theta) - \mathbf{\Sigma}_n^x(\theta)$ and consider the system of equations

$$\mathbf{b}(\theta)\mathbf{S}^1(\theta) = 0, \tag{28}$$

with $\mathbf{b}(\theta) = (\, b_1(\theta) \quad b_2(\theta) \quad \cdots \quad b_n(\theta) \,)$. The subset of $[-\pi, \pi]$ where $\text{rank}(\mathbf{S}^1(\theta)) = 0$, call it M_0, is measurable, possibly empty. In M_0 put $\mathbf{p}_{n1}^x(\theta) = (\, 1 \quad 1 \quad \cdots \quad 1 \,)/\sqrt{n}$. Let M_1 be the measurable subset of $[-\pi, \pi]$ where $\text{rank}(\mathbf{S}^1(\theta)) = 1$. Then let M_1^1 be the measurable subset of M_1 where $S_{11}^1(\theta) \neq 0$. Put $b_2(\theta) = 1$, $b_j(\theta) = 0$ for $j > 2$, then obtain the unique solution for $b_1(\theta)$, i.e. $b_1(\theta) = S_{21}^1(\theta)/S_{11}(\theta)$. Putting $\mathbf{p}_{n1}^x(\theta) = \mathbf{b}(\theta)/|\mathbf{b}(\theta)|$ in M_1^1 we have a measurable function in M_1^1 (notice that the choice $-\mathbf{b}(\theta)/|\mathbf{b}(\theta)|$ would be also valid). Now consider the subset of $M_1 - M_1^1$ where $S_{22}^1(\theta) \neq 0$, and repeat the construction by taking $b_1(\theta) = 1$, $b_j(\theta) = 0$ for $j > 2$. It is clear how to proceed to cover M_1. Then consider M_2, the set where $\text{rank}(\mathbf{S}^1(\theta)) = 2$, and so on until M_{n-1}, the subset of $[-\pi, \pi]$ where $\text{rank}(\mathbf{S}^1(\theta)) = n - 1$. Define the subset M_1^{n-1} as that in which the top-left submatrix of order $n - 1$ of $\mathbf{S}^1(\theta)$ is non singular. Put $b_n(\theta) = 1$ in M_1^{n-1} and find the unique solution to (28). Then again put $\mathbf{p}_{n1}^x(\theta) = \mathbf{b}(\theta)/|\mathbf{b}(\theta)|$. Continuing in this way until M^{n-1} has been covered we obtain a measurable function for $\mathbf{p}_{n1}^x$ defined on $[-\pi, \pi]$. Now consider $\mathbf{S}^2(\theta) = \mathbf{I}_n \lambda_{n2}^x(\theta) - \mathbf{\Sigma}_n^x$ and the system

$$\mathbf{b}(\theta)\mathbf{S}^2(\theta) = 0$$
$$\mathbf{b}(\theta)\tilde{\mathbf{p}}_{n1}^x(\theta) = 0. \tag{29}$$

The procedure above can be applied to (29) so that we obtain a measurable $\mathbf{p}_{n2}^x$ fulfilling (1) and (2). Iterating we reach the last step, when the system is

$$\mathbf{b}(\theta)\mathbf{S}^{n-1}(\theta) = 0$$
$$\mathbf{b}(\theta)\tilde{\mathbf{p}}_{nj}^x(\theta) = 0 \text{ for } j \leq n - 1.$$

QED

Proof of Lemma 7. We have

$$|\mathcal{S}(A_n, B_n; \theta) - \mathcal{S}(A, B; \theta)|$$

$$\leq |\mathcal{S}(A_n, B_n; \theta) - \mathcal{S}(A_n, B; \theta)| + |\mathcal{S}(A_n, B; \theta) - \mathcal{S}(A, B; \theta)|$$

$$= |\mathcal{S}(A_n, B_n - B; \theta)| + |\mathcal{S}(A_n - A, B; \theta)|$$

$$\leq \sqrt{\mathcal{S}(A_n, A_n; \theta)}\sqrt{\mathcal{S}(B_n - B, B_n - B; \theta)} + \sqrt{\mathcal{S}(B, B; \theta)}\sqrt{(A_n - A, A_n - A; \theta)}$$

$$\leq \left[\sqrt{\mathcal{S}(A, A; \theta)} + \sqrt{\mathcal{S}(A_n - A, A_n - A; \theta)}\right]\sqrt{\mathcal{S}(B_n - B, B_n - B; \theta)}$$

$$+ \sqrt{\mathcal{S}(B, B; \theta)}\sqrt{\mathcal{S}(A_n - A, A_n - A; \theta)}.$$

Since $\mathcal{S}(A_n - A, A_n - A; \theta)$ and $\mathcal{S}(B_n - B, B_n - B; \theta)$ converge to zero in the mean, by Fact L_2 there exists a sequence s_i such that $\mathcal{S}(A_{s_i} - A, A_{s_i} - A; \theta)$ and $\mathcal{S}(B_{s_i} - B, B_{s_i} - B; \theta)$ converge to zero a.e. in $[-\pi, \pi]$. **QED**

Fact S. *If $y_t \in \mathbf{X}$, then y_t has a spectral density.*
Sketch of the proof. Let

$$y_t = \lim_n \sum_{j=1}^{n} a_{nj}(L)x_{jt}, \tag{30}$$

let f_n be the spectral density of $\sum_{j=1}^{n} a_{nj}(L)x_{jt}$ and g_{nm} the spectral density of

$$\sum_{j=1}^{n} a_{nj}(L)x_{jt} - \sum_{j=1}^{m} a_{mj}(L)x_{jt}.$$

Equation (30) implies that g_{nm} converges to zero in mean as n and m tend to infinity. The same argument employed in Lemma 7 leads to the conclusion that f_n converges in the mean, call f the limit. Proving that f actually is the spectral density of y_t is not difficult but rather tiresome. The interested reader may request the proof from the authors.

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