

Combining the Theory of Evidence with Fuzzy Sets for Binomial Option Pricing

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ABSTRACT

The aim of this paper is to price an option, in a one-period binomial model, written on a stock whose possible jumps are opaque to the investors. The opacity is captured by the use of fuzzy sets. The pricing methodology is still based on a risk neutral valuation approach, whereby weighted intervals of risk neutral probabilities, are used. These intervals of probabilities arise because of the uncertainty on the magnitude of the two possible states up and down of the binomial tree, even if the real probabilities of the stock price jumps are crisp and known in advance.

The idea of using intervals of risk neutral probabilities instead of a point estimate goes back to the Dempster and Shafer theory of evidence that is based upon two types of nonadditive measures: belief and plausibility measures. Since belief measures are always smaller than or equal to the corresponding plausibility measures, they may be seen as lower and upper probabilities respectively. Computing the expected value of the call under the risk neutral probabilities results in an expected value interval, within which we do not know what is the most likely price.

Our methodology is able to overcome this limit. Using weighted intervals of probabilities, i.e. possibility distributions on the risk neutral probabilities, we find a weighted expected value interval for the call price and thus we are able to determine a “most likely” value of the call within the interval.

Keywords: Evidence theory, Fuzzy sets, Options, Pricing.

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1. INTRODUCTION

The additivity property of classical probability measure is too restrictive in some application contexts, such as finance, where risk is replaced by uncertainty. Additivity works well under error free conditions, but it doesn't seem appropriate for real, physical measurements, where errors, the presence of non-repeatable experiments or the need of subjective judgements complicates the idealized setting. Information ambiguity leaves room to grey areas in which is impossible to give precise probability estimates. Unless using additional assumptions, classical probability theory is incapable of accounting for this type of uncertainty.

Sometimes reliable estimates of the risk neutral probabilities involved are hard to come by. When it is impossible to assess complete probability distributions, one can try to bound the acceptable probabilities of events, working with intervals rather than precise point estimates.

This is the essence of the Dempster and Shafer theory of evidence that is based upon two types of nonadditive measures: belief and plausibility measures. These measures are obtained by replacing the additivity requirement by superadditivity or subadditivity respectively. The dual relationship between the two types of measures ensures that given a measure of either of the two types, it induces a unique measure of the other type. Since belief measures are always smaller than or equal to the corresponding plausibility measures, they may be seen as lower and upper probabilities respectively. Computing the expected value of the call under the risk neutral probabilities results in an expected value interval, within which we do not know what is the most likely price.

In this paper we use weighted intervals of probabilities, i.e. possibility distributions on the risk neutral probabilities, obtaining a weighted expected value interval and thus a "most likely" value of the call within the interval. We price a call option, written on an underlying in a simple one period binomial model. The pricing methodology is based on the no-arbitrage principle.

The purpose of this paper is to trace back the necessity of using an interval of probabilities, to the opacity of the two possible states up and down of the binomial tree. In our setting the risk neutral probabilities interval arise from the possibility distributions given on each of the two possible values of the asset.

The plan of the paper is the following. In section 2 we describe the opacity in the two possible jumps by the use of fuzzy triangular numbers. In section 3 we derive the risk neutral probabilities and we analyse their main characteristics. In section 4 we present the payoff of an option written on the stock and in section 5 we describe the pricing methodology. In section 6 we price a put option written on the stock and in the last section we provide conclusions and some lines of future research work.

2. THE FUZZY-BINOMIAL TREE

We assume that the price of the underlying at $t=1$ takes only two possible values: given the current value P_0^R it may either jump up or down with an exogenously given probability p and $(1-p)$ $p \in [0,1]$. The fuzziness present in the model is of the following type: since we do not know the exact increase or decrease in the stock price, the two possible events up and down are ill-defined. We thus have two possibility distributions: one for the increase and one for the decrease of the stock price, as illustrated in Figure 1. Triangular fuzzy numbers are used to model the two possibility distributions. Among all the different types of numbers, the choice of using triangular numbers is made for the sake of simplicity, since assuming more complicated shapes may increase the computational complexity without substantially affecting the significance of the results. The up and down jump factors, u and d respectively, are represented by two triangular fuzzy numbers identified by the following characteristic function $\mu_i(x)$, linear in x :

$$\mu_i(x) = \begin{cases} 0, & 0 \leq x \leq i_1; \\ \frac{x - i_1}{i_2 - i_1}, & i_1 \leq x \leq i_2; \\ \frac{x - i_3}{i_2 - i_3}, & i_2 \leq x \leq i_3; \\ 0, & x \geq i_3 \end{cases}$$

where $x \in X$, $0 \leq i_1 \leq i_2 \leq i_3$, $i = \{d, u\}$.

More simply, for each state, we can write: $I = (i_1, i_2, i_3)$, where i_1 is the minimum possible value, i_3 the maximum, and i_2 the most possible. The possibility distribution is induced by the characteristic function of the fuzzy set.

Alternatively, we can write a triangular fuzzy number in terms of its α -cuts (or confidence intervals) by the following formula:

$$i(\alpha) = [i_1(\alpha), i_3(\alpha)] = [i_1 + \alpha(i_2 - i_1), i_3 - \alpha(i_3 - i_2)]$$

where α is the level of confidence, $\alpha \in [0,1]$, $i = \{d, u\}$.

This representation will be useful to do some algebra with fuzzy numbers.

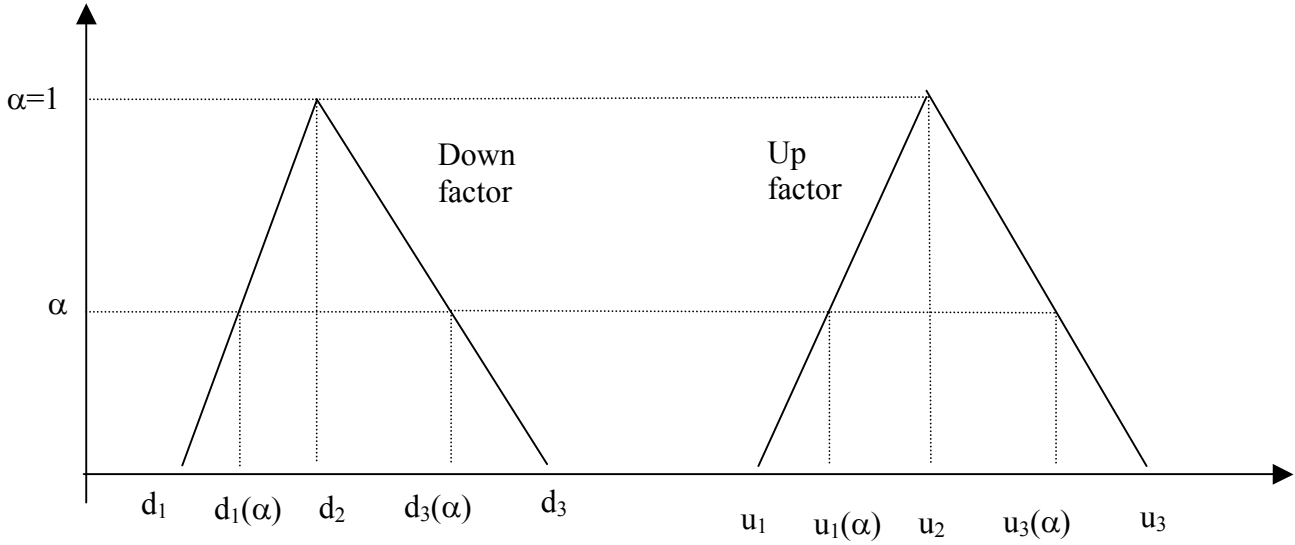


Figure 1. The two possible jump factors of the underlying at $t=1$.

3. THE RISK NEUTRAL PROBABILITIES INTERVALS

The aim of this section is to derive the risk neutral probabilities in order to price a call written on a stock. Let us consider a one-period model where $t \in [0,1]$ is time, the two basic securities are: the money market account, and the risky stock. The money market account, is worth one at $t=0$ and its value at $t=1$ is $1+r$, where r is the risk-free interest rate. The stock price at time zero, P_0^R , is observable, while its price at time one, is obtained multiplying P_0^R by the jump factors introduced in the previous section.

The following assumptions are made:

A1) All investors have homogeneous beliefs.

A2) Markets are frictionless i.e. markets have no transaction costs, no taxes, no restrictions on short sales and asset are infinitely divisible.

A3) Every investor acts as a price taker.

A4) Interest rates are positive. The interest rate is equal to $r > 0$ percent per unit time.

A5) No arbitrage opportunities are allowed. This condition is expressed by the following formula:

$$d_3 < (1+r) < u_1 \quad (1)$$

A6) The market is incomplete $\forall \alpha \in [0,1[$.

On the basis of these assumptions we now apply the standard methodology for deriving the risk neutral probabilities.

Starting from the system:

$$\begin{bmatrix} 1 & 1 \\ \frac{P_0^R d}{1+r} & \frac{P_0^R u}{1+r} \end{bmatrix} \begin{bmatrix} p_d \\ p_u \end{bmatrix} = \begin{bmatrix} 1 \\ P_0^R \end{bmatrix}$$

we derive the two following equations:

$$\begin{cases} p_d + p_u = 1 \\ \frac{d}{1+r} p_d + \frac{u}{1+r} p_u = 1 \end{cases}$$

writing u and d in terms of α -cuts yields:

$$\begin{cases} p_d + p_u = 1 \\ \frac{[d_1^\alpha, d_3^\alpha]}{1+r} p_d + \frac{[u_1^\alpha, u_3^\alpha]}{1+r} p_u = [1,1] \end{cases}$$

This system may be splitted in the following two:

$$\begin{cases} p_d + p_u = 1 \\ \frac{d_1 + \alpha(d_2 - d_1)}{1+r} p_d + \frac{u_1 + \alpha(u_2 - u_1)}{1+r} p_u = 1 \end{cases} \quad (2)$$

$$\begin{cases} p_d + p_u = 1 \\ \frac{d_3 - \alpha(d_3 - d_2)}{1+r} p_d + \frac{u_3 - \alpha(u_3 - u_2)}{1+r} p_u = 1 \end{cases} \quad (3)$$

Solving system (2) yields:

$$\begin{cases} p_u = \frac{(1+r) - d_1 - \alpha(d_2 - d_1)}{u_1 - d_1 + \alpha(u_2 - u_1 - d_2 + d_1)} \\ p_d = \frac{u_1 + \alpha(u_2 - u_1) - (1+r)}{u_1 - d_1 + \alpha(u_2 - u_1 - d_2 + d_1)} \end{cases}$$

solving system (3) yields:

$$\begin{cases} p_u = \frac{(1+r) - d_3 + \alpha(d_3 - d_2)}{u_3 - d_3 - \alpha(u_3 - u_2 - d_3 + d_2)} \\ p_d = \frac{u_3 - \alpha(u_3 - u_2) - (1+r)}{u_3 - d_3 - \alpha(u_3 - u_2 - d_3 + d_2)} \end{cases}$$

The two solutions represent the bounds of the intervals of probabilities that are respectively:

$$\begin{aligned}
p_u &= [\underline{p}_u, \bar{p}_u] = \left[\frac{(1+r) - d_3 + \alpha(d_3 - d_2)}{u_3 - d_3 - \alpha(u_3 - u_2 - d_3 + d_2)}, \frac{(1+r) - d_1 - \alpha(d_2 - d_1)}{u_1 - d_1 + \alpha(u_2 - u_1 - d_2 + d_1)} \right] \\
p_d &= [\underline{p}_d, \bar{p}_d] = \left[\frac{u_1 + \alpha(u_2 - u_1) - (1+r)}{u_1 - d_1 + \alpha(u_2 - u_1 - d_2 + d_1)}, \frac{u_3 - \alpha(u_3 - u_2) - (1+r)}{u_3 - d_3 - \alpha(u_3 - u_2 - d_3 + d_2)} \right]
\end{aligned} \tag{4}$$

It is easy to check that the following duality relations hold: $\bar{p}_u + \underline{p}_d = 1$ and $\bar{p}_d + \underline{p}_u = 1$. To draw a comparison with Evidence Theory, we indeed have two measures, \bar{p}_i and \underline{p}_i , with $i=d,u$, where \bar{p}_i is the dual measure of \underline{p}_i .

It is interesting to observe that, differently from the standard binomial option pricing model, we obtain risk neutral probability intervals instead of point estimates. This is clearly a consequence of the incompleteness of the market (except for $\alpha=1$). The risk neutral probability intervals arise from the opacity of the stock price at $t=1$, even if the real probabilities of the stock price jumps are crisp and known in advance.

Moreover, the intervals of risk neutral probabilities are weighted, they are indeed fuzzy numbers. This is a very important feature of our pricing model, since it allows us to find a weighted expected value interval for the call price as is shown in the following sections.

In order to determine the shape of the two probabilities, we compute their value at $\alpha=0$ and $\alpha=1$ and then we analyse their behaviour as α varies (proofs are in Appendix 3).

If $\alpha=0$ then:

$$\begin{aligned}
p_u &= \left[\frac{(1+r) - d_3}{u_3 - d_3}, \frac{(1+r) - d_1}{u_1 - d_1} \right] \\
p_d &= \left[\frac{u_1 - (1+r)}{u_1 - d_1}, \frac{u_3 - (1+r)}{u_3 - d_3} \right]
\end{aligned} \tag{5}$$

If $\alpha=1$ then:

$$\begin{aligned}
p_u &= \frac{(1+r) - d_2}{u_2 - d_2} \\
p_d &= \frac{u_2 - (1+r)}{u_2 - d_2}
\end{aligned} \tag{6}$$

It is easy to show that the derivative with respect to α is positive for both the left bounds and is negative for both the right bounds of the probabilities. This means that the bounds are getting narrower as α increases. In particular, if $\alpha=1$, these bounds collapse in only one point. If $\alpha=1$, the stock price in each state assume only one value, in other words, the market is complete and as a consequence we find a unique risk neutral probability measure. Thus our model can be seen as a

generalisation of the standard binomial option pricing model as the latter is a special case (if $\alpha=1$) of the former.

By inspection of \underline{p}_u , it is easy to prove that its first derivative is positive and that the second derivative is positive if $u_3 - u_2 > d_3 - d_2$; in the opposite case it is negative. Note that if $u_3 - u_2 = d_3 - d_2$ then \underline{p}_u is linear in α .

Analogously it is easy to prove that the first derivative of \overline{p}_u is negative and that the second derivative is positive if $u_2 - u_1 > d_2 - d_1$; in the opposite case it is negative. Note that if $u_2 - u_1 = d_2 - d_1$ then \overline{p}_u is linear in α .

As for \underline{p}_d , we can prove that its first derivative is positive and that the second derivative is negative if $u_2 - u_1 > d_2 - d_1$; in the opposite case it is positive. Note that if $u_3 - u_2 = d_3 - d_2$ then \underline{p}_d is linear in α .

Analogously for \overline{p}_d , we can prove that its first derivative is negative and that the second derivative is negative if $u_3 - u_2 > d_3 - d_2$; in the opposite case it is positive. Note that if $u_3 - u_2 = d_3 - d_2$ then \overline{p}_d is linear in α .

It follows that depending on the relative positions of $u_1, u_2, u_3, d_1, d_2, d_3$ we can have different shapes for p_u and p_d as illustrated in Table 1. The graphs, that are just possible outcomes, show how the probability intervals shrink with α . In fact for $\alpha=1$ each of the risk neutral probabilities assumes a single value. Note that the two bounds and the most possible value are determined by equations (5) and (6).

In table 1 are not reported, for reasons of space, the cases in which we have $u_2 - d_2 = u_1 - d_1$ or $u_3 - d_3 = u_2 - d_2$. It is clear that if $u_2 - d_2 = u_1 - d_1$ then both \overline{p}_u and \underline{p}_d are linear in α ; if $u_3 - d_3 = u_2 - d_2$ then both \underline{p}_u and \overline{p}_d are linear in α .

As a special case we examine what happens if both the triangular fuzzy numbers that represent the up and down jump factors are symmetrical and equally flat, i.e. if

$$u_3 - u_2 = u_2 - u_1 = d_3 - d_2 = d_2 - d_1 = k, \quad (7)$$

k being the left or right spread.

Note that this implies also:

$$u_3 - d_3 = u_2 - d_2 = u_1 - d_1 = h$$

In this case both p_u and p_d are linear in α , i.e. they are triangular fuzzy numbers:

$$\begin{aligned}
p_u &= [\underline{p}_u, \bar{p}_u] = \left[\frac{(1+r) - d_3 + \alpha k}{h}, \frac{(1+r) - d_1 - \alpha k}{h} \right] \\
p_d &= [\underline{p}_d, \bar{p}_d] = \left[\frac{u_1 + \alpha k - (1+r)}{h}, \frac{u_3 - \alpha k - (1+r)}{h} \right]
\end{aligned}
\tag{8}$$

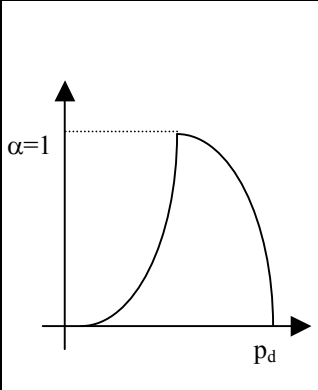
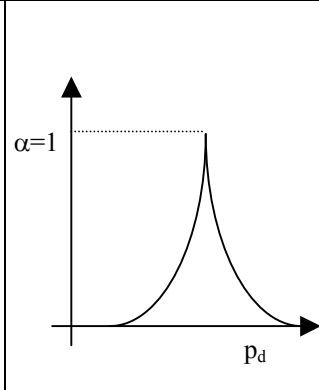
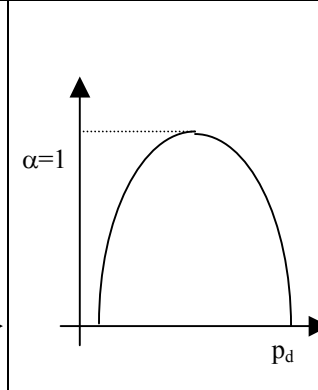
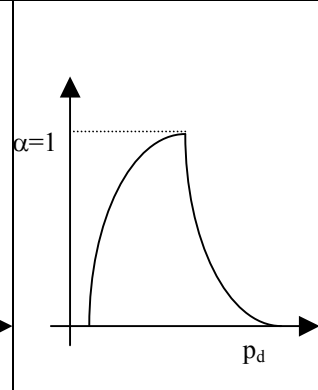
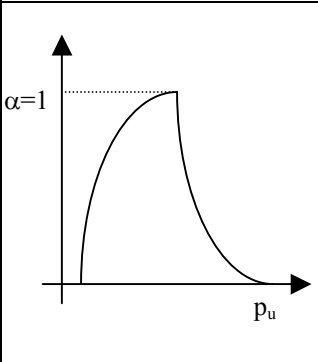
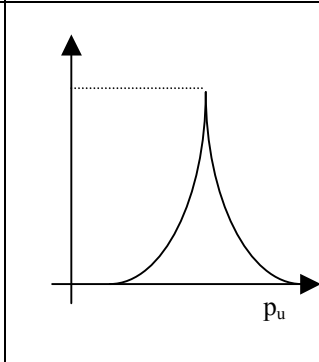
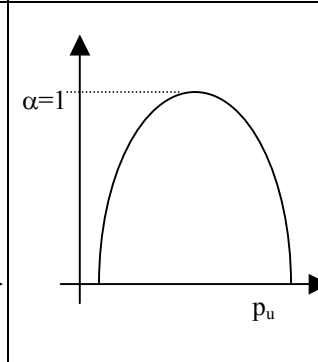
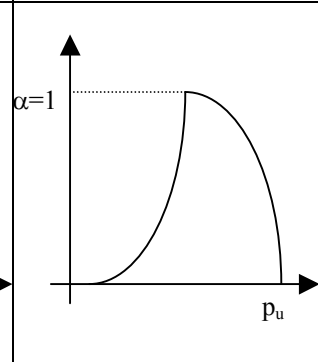
	$u_2 - u_1 > d_2 - d_1$ $u_3 - u_2 > d_3 - d_2$	$u_2 - u_1 > d_2 - d_1$ $u_3 - u_2 < d_3 - d_2$	$u_2 - u_1 < d_2 - d_1$ $u_3 - u_2 > d_3 - d_2$	$u_2 - u_1 < d_2 - d_1$ $u_3 - u_2 < d_3 - d_2$
p_d				
p_u				

Table 1. The possible shapes of the artificial probabilities.

4. THE PAYOFF OF A CALL OPTION

At the maturity date, a call option has a positive value if the price of the underlying is greater than the exercise price; in the opposite case it remains unexercised and has zero value. As we are in a one period model, it makes no sense to distinguish between American and European options.

The payoff of a call option depends on the price of the underlying asset. The stock price at $t=1$ is given by either $P_0^R d$ or $P_0^R u$. Since u and d are triangular fuzzy numbers, it follows that the stock price at $t=1$ in each state is represented by a triangular fuzzy number, as represented in figure 2.

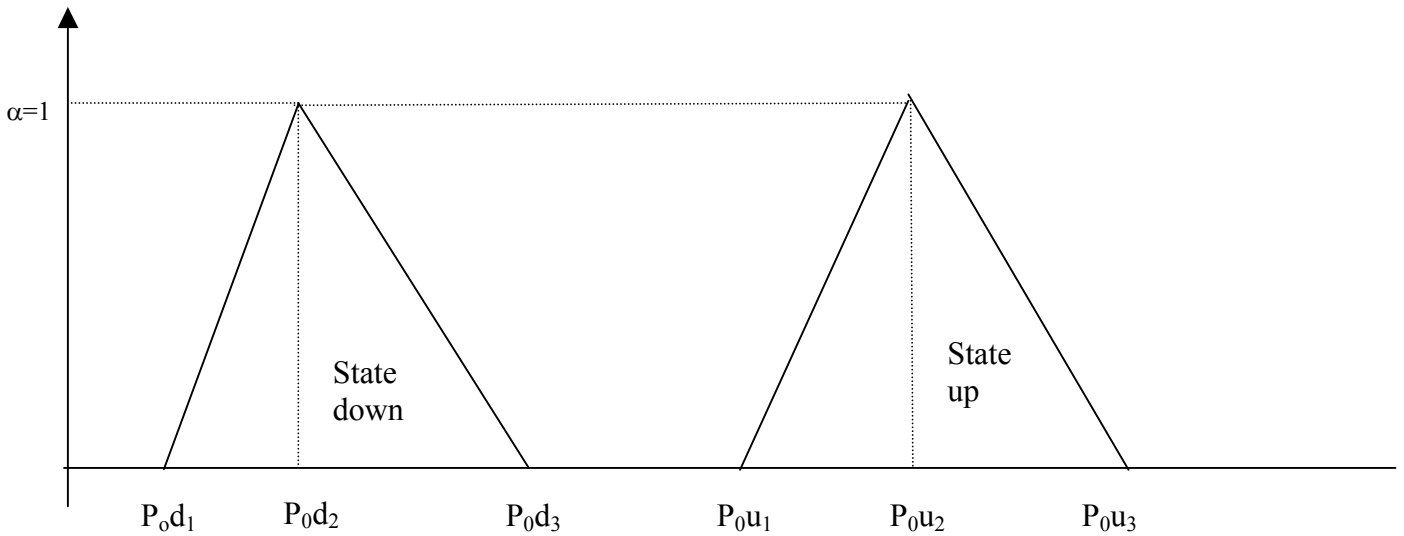


Figure 2. The stock price at $t=1$.

To make an option an interesting contract we assume the following condition on the strike price:

$$P_0d_3 \leq X \leq P_0u_1$$

We denote the call payoff in state “up” with $C(u)$ and in state down with $C(d)$. It follows that

$$C(d) = 0 \text{ and } C(u) = (P_0u - X).$$

Applying the algebra of fuzzy numbers (see Appendix 2), we obtain the call payoff, which is still a triangular fuzzy number equal to:

$$C(u) = (P_0u_1 - X, P_0u_2 - X, P_0u_3 - X)$$

as shown in figure 3.

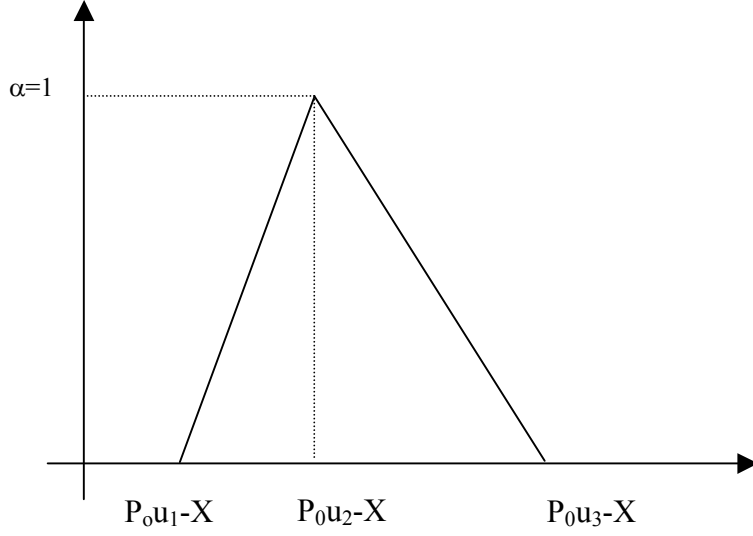


Figure 3. The Call payoff in the state “up”.

5. THE PRICING METHODOLOGY

We now determine the call price C_0 by means of the risk neutral valuation approach, as follows:

$$C_0 = \frac{1}{1+r} \hat{E}[C_1] = \frac{1}{1+r} [p_d * C(d) + p_u * C(u)]$$

where \hat{E} stands for expectation under the risk neutral probabilities and C_1 is the payoff of the call at $t=1$.

Since the call has zero payoff in the down state, the option price formula simplifies to:

$$C_0 = \frac{1}{1+r} [p_u * C(u)]$$

Following the rules of multiplication between fuzzy numbers, reported in Appendix 2, as we have an interval for p_u :

$$p_u = [\underline{p}_u, \bar{p}_u] = \left[\frac{(1+r) - d_3 + \alpha(d_3 - d_2)}{u_3 - d_3 - \alpha(u_3 - u_2 - d_3 + d_2)}, \frac{(1+r) - d_1 - \alpha(d_2 - d_1)}{u_1 - d_1 + \alpha(u_2 - u_1 - d_2 + d_1)} \right]$$

we have also an interval for the call prices.

$$C_0 = [\underline{C}_0, \bar{C}_0] = \left[\frac{P_0 u_1 - X + \alpha P_0 (u_2 - u_1)}{1+r} * \frac{(1+r) - d_3 + \alpha(d_3 - d_2)}{u_3 - d_3 - \alpha(u_3 - u_2 - d_3 + d_2)}, \frac{P_0 u_3 - X - \alpha P_0 (u_3 - u_2)}{1+r} * \frac{(1+r) - d_1 - \alpha(d_2 - d_1)}{u_1 - d_1 + \alpha(u_2 - u_1 - d_2 + d_1)} \right] \quad (9)$$

It is easy to prove that as α increases the call option interval of prices shrinks (for the proof see Appendix 4). It follows that if $\alpha=0$ the price interval is the largest:

$$C_0 = \frac{1}{1+r} \left[(P_0 u_1 - X) * \frac{(1+r) - d_3}{u_3 - d_3}, (P_0 u_3 - X) * \frac{(1+r) - d_1}{u_1 - d_1} \right]. \quad (10)$$

If $\alpha=1$, the market is complete and the interval collapses into only one value:

$$C_0 = \frac{P_0 u_2 - X}{1+r} * \frac{(1+r) - d_2}{u_2 - d_2}. \quad (11)$$

As expected, this is the same result of the standard binomial option pricing model.

Having a weighted expected value interval for the call price is clearly a very important feature for financial applications since it enables us to determine the most possible outcome of the call price.

It is also interesting to observe that the “most likely value” of the call is the one that we would have obtained if the market were complete. In our model the incompleteness of the market that arise because of the opacity in the possible jumps of the underlying provides us with an interval of call prices built around the “complete market” price.

Analysing the shape of the call option price we note that the left part:

$$\underline{C}_0 = \frac{P_0 u_1 - X + \alpha P_0 (u_2 - u_1)}{1+r} * \frac{(1+r) - d_3 + \alpha(d_3 - d_2)}{u_3 - \alpha(u_3 - u_2) - d_3 + \alpha(d_3 - d_2)}$$

is increasing in α and is concave or convex depending on the sign of the quantity $(u_3 - d_3)(P_0 u_2 - X) - (u_2 - d_2)(P_0 u_1 - X)$ as illustrated in Table 2 (for the proofs see Appendix 4).

Note that if $(u_3 - d_3)(P_0 u_2 - X) = (u_2 - d_2)(P_0 u_1 - X)$, then is linear in α .

The right part:

$$\overline{C}_0 = \frac{P_0 u_3 - X - \alpha P_0 (u_3 - u_2)}{1+r} * \frac{(1+r) - d_1 - \alpha(d_2 - d_1)}{u_1 - d_1 + \alpha(u_2 - u_1 - d_2 + d_1)}$$

is increasing in α and is concave or convex depending on the sign of the quantity $(u_2 - d_2)(P_0 u_3 - X) - (u_1 - d_1)(P_0 u_2 - X)$ as illustrated in Table 2 (for the proofs see Appendix 4).

Note that if $(u_2 - d_2)(P_0 u_3 - X) = (u_1 - d_1)(P_0 u_2 - X)$, then is linear in α .

Note that the two bounds and the most possible value are determined by equations (10) and (11).

The graphs in Table 2 are just possible outcomes. Cases in which the call price is linear in α are not reported in the table.

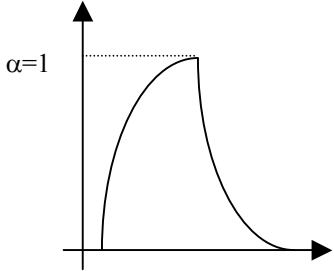
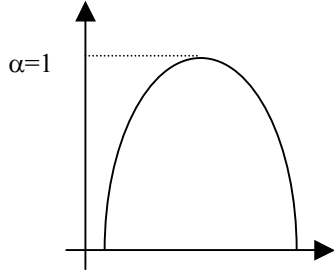
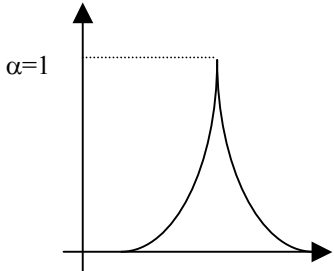
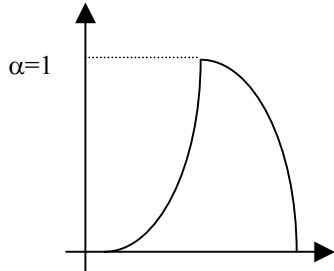
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Table 2. Possible shapes of the call option price.

6. THE PRICING OF A PUT OPTION

At the maturity date, a put option has a positive value if the price of the underlying is smaller than the exercise price K ; in the opposite case it has zero value and remains unexercised. As we are in a one period model, it makes no sense to distinguish between American and European options.

To make a put option an interesting contract we assume that:

$$P_0 d_3 \leq K \leq P_0 u_1$$

We denote the put payoff in state “up” with $Q(u)$ and in state down with $Q(d)$. It follows that $Q(d) = (K - P_0 d)$ and $Q(u) = 0$.

Applying the algebra of fuzzy numbers (see Appendix 2), we obtain the put payoff which is still a triangular fuzzy number equal to:

$$Q(d) = (K - P_0 d_3, K - P_0 d_2, K - P_0 d_1)$$

as shown in figure 4.

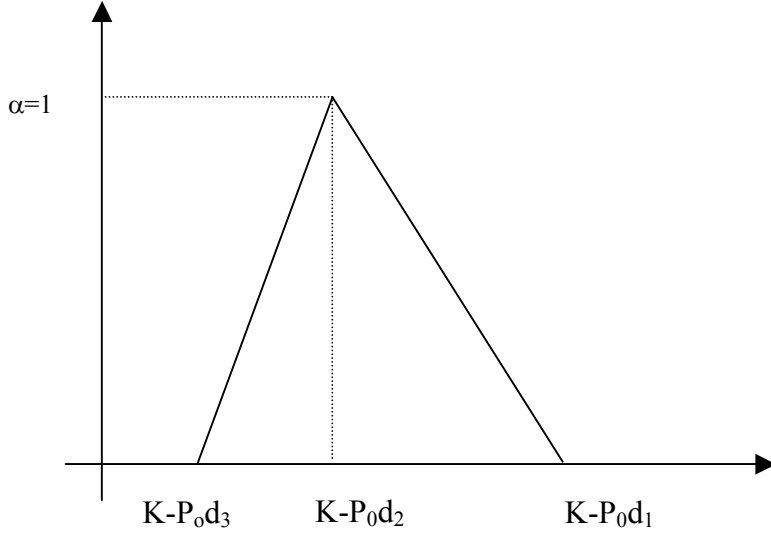


Figure 4. The Put payoff in the state “down”.

We now determine the put price Q_0 by means of the risk neutral valuation approach, as follows:

$$Q_0 = \frac{1}{1+r} \hat{E}[Q_1] = \frac{1}{1+r} [p_d * Q(d) + p_u * Q(u)]$$

where \hat{E} stands for expectation under the risk neutral probabilities and Q_1 is the payoff of the put at $t=1$. Since the put has zero payoff in state up, the option price formula simplifies to:

$$Q_0 = \frac{1}{1+r} [p_d * Q(d)]$$

Following the rules of multiplication between fuzzy numbers, reported in Appendix 2, as we have an interval for p_d :

$$p_d = [\underline{p}_d, \bar{p}_d] = \left[\frac{u_1 + \alpha(u_2 - u_1) - (1+r)}{u_1 - d_1 + \alpha(u_2 - u_1 - d_2 + d_1)}, \frac{u_3 - \alpha(u_3 - u_2) - (1+r)}{u_3 - d_3 - \alpha(u_3 - u_2 - d_3 + d_2)} \right]$$

we have also an interval for the put prices.

$$Q_0 = [\underline{Q}_0, \bar{Q}_0] = \left[\frac{K - P_0d_3 + \alpha P_0(d_3 - d_2)}{1+r} * \frac{u_1 + \alpha(u_2 - u_1) - (1+r)}{u_1 - d_1 + \alpha(u_2 - u_1 - d_2 + d_1)}, \frac{K - P_0d_1 - \alpha P_0(d_2 - d_1)}{1+r} * \frac{u_3 - \alpha(u_3 - u_2) - (1+r)}{u_3 - d_3 - \alpha(u_3 - u_2 - d_3 + d_2)} \right] \quad (12)$$

It is easy to proof that as α increases the put option interval of prices shrinks (for the proof see the appendix). It follows that if $\alpha=0$ the price interval is the largest:

$$Q_0 = \frac{1}{1+r} \left[(K - P_0d_3) * \frac{u_1 - (1+r)}{u_1 - d_1}, (K - P_0d_1) * \frac{u_3 - (1+r)}{u_3 - d_3} \right]. \quad (13)$$

If $\alpha=1$ the price interval collapses to:

$$Q_0 = \frac{K - P_0 d_2}{1 + r} * \frac{u_2 - (1 + r)}{u_2 - d_2}. \quad (14)$$

7. CONCLUSIONS

In this paper we have priced an option, in a one-period binomial model, written on a stock whose possible jumps are opaque to the investors, being modelled by the use of triangular fuzzy numbers. The pricing methodology is still based on a risk neutral valuation approach, whereby weighted intervals of risk neutral probabilities, are used. These intervals of probabilities arise because of the uncertainty on the magnitude of the two possible states up and down of the binomial tree, even if the real probabilities of the stock price jumps are crisp and known in advance. The possibility distribution given on each of the two possible jump of the asset induces in turn a possibility distribution on each of the risk neutral probabilities. It follows that we have also a weighted expected value interval for the call price, where the best guess is the call price computed supposing that the market is complete.

Our methodology offers some advantages. First, it provides an intuitive way to look at the uncertainty in the stock price jumps. Second, it includes the results of the Standard Binomial Option Pricing Model. Third, it traces back the need of using intervals of risk neutral probabilities, to the opacity in the two possible jumps of the stock. Finally, using weighted intervals of probabilities, i.e. possibility distributions on the risk neutral probabilities, it provides us with a weighted expected value interval for the call price and thus we are able to determine a “most likely” value of the call within the interval.

This work has to be seen as preliminary to future research and still lends itself to be extended in many directions. High on the research agenda are the extension to different shapes of fuzzy numbers that represent the two jumps of the stock price and to a multiperiod discrete version of the model.

Appendix 1. Belief measures and plausibility measures.

Let $P(P(X))$ be the power set of $P(X)$. If p is a discrete probability measure on $(P(X), P(P(X)))$, with $p(\{\emptyset\})=0$, then the set function $m: P(X) \rightarrow [0,1]$ determined by:

$$m(E) = p(\{E\}) \text{ for any } E \in P(X)$$

is called a basic probability assignment on $P(X)$.

A set function $m: P(X) \rightarrow [0,1]$ is called a basic probability assignment if and only if:

- 1) $m(\emptyset) = 0$
- 2) $\sum_{E \in P(X)} m(E) = 1$

for the proof see [11] p.55.

If m is basic probability assignment on $P(X)$, then the set function $Bel: P(X) \rightarrow [0,1]$ determined by:

$$Bel(E) = \sum_{F \subset E} m(F) \text{ for any } E \in P(X)$$

Is called a belief measure on $(X, P(X))$ induced from m .

Theorem 1). If Bel is a belief measure on $(X, P(X))$, then:

- 1) $Bel(\emptyset) = 0$
- 2) $Bel(X) = 1$
- 3) $Bel\left(\bigcup_{i=1}^n E_i\right) \geq \sum_{I \subset \{1, \dots, n\}, I \neq \emptyset} (-1)^{|I|+1} Bel \bigcap_{i \in I} E_i$
- 4) Bel is continuous from above.
- 5) Bel is monotone and superadditive.

for the proofs see [11] pp.56-58.

If m is basic probability assignment on $P(X)$, then the set function $Pl: P(X) \rightarrow [0,1]$ determined by:

$$Pl(E) = \sum_{F \cap E \neq \emptyset} m(F) \text{ for any } E \in P(X)$$

Is called a plausibility measure on $(X, P(X))$ induced from m .

Theorem 2). If Pl is a plausibility measure on $(X, P(X))$, then:

- 1) $Pl(\emptyset) = 0$
- 2) $Pl(X) = 1$
- 3) $Pl\left(\bigcap_{i=1}^n E_i\right) \leq \sum_{I \subset \{1, \dots, n\}, I \neq \emptyset} (-1)^{|I|+1} Pl\left(\bigcup_{i \in I} E_i\right)$
- 4) Pl is continuous from below.
- 5) Pl is monotone and subadditive.

for the proofs see [11] pp.60-61.

Theorem 3). If Bel and Pl are the belief measures and plausibility measures induced from the same basic probability assignment, then:

$$\text{Bel}(E)=1-\text{Pl}(\bar{E})$$

$$\text{Bel}(E)\leq\text{Pl}(E)$$

for the proofs see [11] p.59.

Theorem 4). If Bel coincides with Pl, then m focuses only on singletons.

for the proofs see [11] p.59.

Appendix 2. Operations on triangular fuzzy numbers.

Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ be two triangular fuzzy numbers written in triplet form, the following operations are defined:

Addition.

$$A + B = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

Subtraction.

$$A - B = (a_1, a_2, a_3) - (b_1, b_2, b_3) = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$$

Multiplication.

$$A * B = (a_1, a_2, a_3) * (b_1, b_2, b_3) = (a_1 * b_1, a_2 * b_2, a_3 * b_3)$$

Division.

$$A : B = (a_1, a_2, a_3) : (b_1, b_2, b_3) = (a_1 : b_3, a_2 : b_2, a_3 : b_1)$$

Multiplication of a triangular fuzzy number by an ordinary number.

$$\forall y \in R, y * A = (y * a_1, y * a_2, y * a_3)$$

Alternatively, a more accurate method of adding fuzzy numbers consists of adding the confidence intervals at each α -level: let $A = [a_1(\alpha), a_3(\alpha)]$, $B = [b_1(\alpha), b_3(\alpha)]$

Addition.

$$A + B = (a_1(\alpha) + b_1(\alpha), a_3(\alpha) + b_3(\alpha))$$

Subtraction.

$$A - B = (a_1(\alpha) - b_3(\alpha), a_3(\alpha) - b_1(\alpha))$$

Multiplication.

$$A * B = (a_1(\alpha) * b_1(\alpha), a_3(\alpha) * b_3(\alpha))$$

Division.

$$A : B = (a_1(\alpha) : b_3(\alpha), a_3(\alpha) : b_1(\alpha))$$

Multiplication of a triangular fuzzy number by an ordinary number.

$$\forall y \in R, y * A = (y * a_1(\alpha), y * a_3(\alpha))$$

Appendix 3. Properties of the risk neutral probabilities.

Recall that the following inequalities hold:

$$d_1 < d_2 < d_3 < (1+r) < u_1 < u_2 < u_3 \quad (A3.1)$$

$$0 < \underline{p}_d < 1, \quad 0 < \bar{p}_d < 1, \quad 0 < \underline{p}_u < 1, \quad 0 < \bar{p}_u < 1 \quad (A3.2)$$

Analysing the behaviour of:

$$\underline{p}_u = \frac{(1+r) - d_3 + \alpha(d_3 - d_2)}{u_3 - d_3 - \alpha(u_3 - u_2 - d_3 + d_2)} = \frac{N}{D}$$

The first derivative with respect to α is:

$$\underline{p}'_u = \frac{(d_3 - d_2)(D - N) + N(u_3 - u_2)}{D^2}$$

Given (A3.1) and (A3.2) the sign is always positive.

The second derivative is:

$$\underline{p}''_u = \frac{+ \underline{p}'_u 2(u_3 - u_2 - d_3 + d_2)}{D}$$

the sign clearly depends on the quantity $(u_3 - u_2 - d_3 + d_2)$, in particular, if $u_3 - u_2 > d_3 - d_2$

the second derivative is positive; in the opposite case it is negative.

Analysing the behaviour of:

$$\bar{p}_u = \frac{(1+r) - d_1 - \alpha(d_2 - d_1)}{u_1 - d_1 + \alpha(u_2 - u_1 - d_2 + d_1)} = \frac{P}{Q}$$

The first derivative with respect to α is:

$$\bar{p}'_u = \frac{-(d_2 - d_1)(Q - P) - P(u_2 - u_1)}{Q^2}$$

Given (A3.1) and (A3.2) the sign is always negative.

The second derivative is:

$$\bar{p}''_u = \frac{- \bar{p}'_u 2(u_2 - u_1 - d_2 + d_1)}{Q}$$

the sign clearly depends on the quantity $(u_2 - u_1 - d_2 + d_1)$, in particular, if $u_2 - u_1 > d_2 - d_1$ the second derivative is positive; in the opposite case it is negative.

Analysing the behaviour of:

$$\underline{p}_d = \frac{u_1 + \alpha(u_2 - u_1) - (1+r)}{u_1 - d_1 + \alpha(u_2 - u_1 - d_2 + d_1)} = \frac{R}{S}$$

The first derivative with respect to α is:

$$\underline{p}'_d = \frac{(u_2 - u_1)(S - R) + R(d_2 - d_1)}{S^2}$$

and is clearly positive, given (A3.1) and (A3.2).

The second derivative is:

$$\underline{p}''_d = \frac{-\underline{p}'_d 2(u_2 - u_1 - d_2 + d_1)}{S}$$

the sign clearly depends on the quantity $(u_2 - u_1 - d_2 + d_1)$, in particular, if $u_2 - u_1 > d_2 - d_1$ the second derivative is negative; in the opposite case it is positive.

Analysing the behaviour of:

$$\bar{p}_d = \frac{u_3 - \alpha(u_3 - u_2) - (1+r)}{u_3 - d_3 - \alpha(u_3 - u_2 - d_3 + d_2)} = \frac{T}{Z}$$

The first derivative with respect to α is:

$$\bar{p}'_d = \frac{-(u_3 - u_2)(Z - T) - (d_3 - d_2)T}{Z^2}$$

and is clearly negative given (A3.1) and (A3.2).

The second derivative is:

$$\bar{p}''_d = \frac{+\bar{p}'_d 2(u_3 - u_2 - d_3 + d_2)}{Z}$$

the sign clearly depends on the quantity $(u_3 - u_2 - d_3 + d_2)$, in particular, if $u_3 - u_2 > d_3 - d_2$ the second derivative is negative; in the opposite case it is positive.

Appendix 4. Properties of the call price.

Recall that the following inequalities hold:

$$d_1 < d_2 < d_3 < (1+r) < u_1 < u_2 < u_3 \quad (\text{A4.1})$$

$$0 < \underline{p}_d < 1, \quad 0 < \bar{p}_d < 1, \quad 0 < \underline{p}_u < 1, \quad 0 < \bar{p}_u < 1 \quad (\text{A4.2})$$

$$P_0 d_3 \leq X \leq P_0 u_1 \quad (\text{A4.3})$$

Analysing the left bound of the call price in equation (9) the first derivative with respect to α is:

$$\underline{C}'_0 = \frac{P_0(u_2 - u_1)}{1+} * \underline{p}_u + \underline{p}'_u \frac{P_0 u_1 - X + \alpha P_0(u_2 - u_1)}{1+} > 0$$

and is positive given A4.1, A4.2 and A4.3 and since \underline{p}'_u is positive (see appendix 3).

The second derivative is:

$$\underline{C}''_0 = \frac{2\underline{p}'_u}{(1+r)[u_3 - d_3 - \alpha(u_3 - u_2 - d_3 + d_2)]} * [(u_3 - d_3)(P_0 u_2 - X) - (u_2 - d_2)(P_0 u_1 - X)]$$

Since the fraction is always positive, the sign depends on the quantity in brackets:

if $[(u_3 - d_3)(P_0 u_2 - X) - (u_2 - d_2)(P_0 u_1 - X)] > 0$ then the sign is positive, in the opposite case it is negative. Note that if $(u_3 - d_3)(P_0 u_2 - X) = (u_2 - d_2)(P_0 u_1 - X)$ then \underline{C}_0 is linear in α .

Analysing the shape of the right part we have that the first derivative with respect to α is:

$$\bar{C}'_0 = -\frac{P_0(u_3 - u_2)}{1+} \bar{p}_u + \bar{p}'_u \frac{P_0 u_3 - X - \alpha P_0(u_3 - u_2)}{1+} < 0$$

and is negative given A4.1, A4.2 and A4.3 and since \bar{p}'_u is negative (see Appendix 3).

The second derivative is:

$$\bar{C}''_0 = -\frac{2\bar{p}'_u}{(1+r)[u_1 - d_1 + \alpha(u_2 - u_1 - d_2 + d_1)]} * [(u_2 - d_2)(P_0 u_3 - X) - (u_1 - d_1)(P_0 u_2 - X)]$$

Since the fraction is always positive, the sign depends on the quantity in brackets:

if $[(u_2 - d_2)(P_0 u_3 - X) - (u_1 - d_1)(P_0 u_2 - X)] > 0$ then the sign is positive, in the opposite case it is negative. Note that if $(u_2 - d_2)(P_0 u_3 - X) = (u_1 - d_1)(P_0 u_2 - X)$ then \bar{C}_0 is linear in α .

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