

Optimisation of a fuzzy non linear function

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Introduction

Fuzzy numbers can be introduced in order to model imprecise situations involving real numbers, and one of the first problem one meets working with these is to decide what type of order to use on the fuzzy number set. In fact this set does not have a natural total ranking. Different methods for ranking fuzzy numbers has been described. Most of these are defined by a function which maps each fuzzy number into an ordered set and transfer the order of one set to the other. We consider that fuzzy numbers may be thought as intervals whose boundaries are blurred, and the difficulty in ranking them arise from the problems created in ranking real intervals. As overlapping is the main difficulty, the problem is overcome when the supports of the fuzzy numbers are disjoint. In this case all the methods give the same solution. By contrast, the decision is not evident when the set intersect. In this case, different methods give different solutions for the same problem. This problem happens even in the classification of real intervals when they are partially overlapped.

The problem we study in this paper is optimising a non-linear function of fuzzy variables with values in the fuzzy number set. At the beginning, we had to start with a definition of ranking fuzzy numbers in order to being able to speak about maximum or minimum of a fuzzy valued function. In two papers ([6] e [7]), Canestrelli and Giove faced an analogous problem. These authors decided to use a definition of “linked variables” to approach the problem of ranking fuzzy numbers. In this paper we use a particular real valued ranking function, called *average value (AV)*, generated by two different ranking functions (*evaluation functions*) on real intervals. The choice has been due to the fact that the AV is defined as dependent on several parameters, allowing flexibility in the final result. Both evaluations functions on real intervals contain a parameter: in the first case it is a real number, in the second it is a function, we call degree of risk, which takes into account of a risk-propension or aversion of the decision maker. The two AV we use are the mean values of the evaluation functions on the α -cuts of the fuzzy numbers obtained by a particular Stieltjes measure generated by a function $s(x) = x^r$ with $r > 0$. We used $r=2$ because this choice gives more weight to the high values of α . In the fourth chapter we produce a necessary and sufficient condition for the existence of a solution of the optimising problem and the interesting result is that it is possible to treat the fuzzy optimisation problem without having any information about the minimum and the maximum of the function. This result should give the opportunity to build an algorithm to reach the solution easier. An interesting future development of the study is the use of AV defined not with additive measures, but with fuzzy measure, using the Choquet’s integral.

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1 Fuzzy numbers

A fuzzy set A is characterized by a generalized characteristic function $\mathbf{m}_A(\cdot)$, called membership function, defined on a Universe X , which assumes values in $[0,1]$. The Universe, in a concrete case, has to be chosen according to the specific situation of the case. In the following $X \subseteq R$.

A fuzzy set is a *fuzzy number* if:

i) $\forall \mathbf{a} \in [0,1]$, the \mathbf{a} -cut of A , $A^{\mathbf{a}} = \{x \in X : \mathbf{m}_A(x) \geq \mathbf{a}\}$ is convex.

ii) $\mathbf{m}_A(\cdot)$ is an upper semicontinuous function.

iii) $\text{supp}(A) = \{x \in X : \mathbf{m}_A(x) > 0\}$ is a bounded set of X

The *height* of A , that is, $\sup_{x \in X} \mathbf{m}_A(x)$ has to be equal to one.

A *triangular fuzzy number TFN* is a fuzzy number defined by a triplet (a_1, a_2, a_3) , such that $\mathbf{m}(x) = 0 \quad \forall x \leq a_1$ and $\forall x \geq a_3$, $\mathbf{m}(a_2) = 1$ and $\mathbf{m}(x)$ is a continuous and piece-linear function $\forall x \in [a_1, a_3]$.

the real line, or the set of fuzzy numbers have not a unique and trivial order. It is possible to define a lot of kinds of orders in both cases, but usually they are not total.

In the next paragraph we define two ranking functions on the real intervals space and on fuzzy numbers.

2 Ranking functions on real intervals and fuzzy numbers.

Def. 2.1. We call “*evaluation function*” a real function $\mathbf{j} : I \rightarrow R$ defined on the family of sets

$$I = \{A = [a_1, a_2] : a_1, a_2 \in R, a_2 \geq a_1\},$$

which depends on the extremal values of the interval

$$\mathbf{j}(A) = \mathbf{j}([a_1, a_2]) = \mathbf{j}(a_1, a_2).$$

In general, it is possible not to request any property on $\mathbf{j}(\cdot)$, but it seems reasonable to consider functions which are increasing in both variables and have some regularity property (i.e. $\mathbf{j} \in C^{(1)}$).

We will consider two types of evaluation functions which are based on a term, called *degree of risk* (the risk-propension or aversion) of the decision maker, that have the very interesting mathematical property to be sensitive to the uncertainty associated to the use of real intervals instead of real numbers.

- the family $\{\mathbf{j}_I\}_{I \in [0,1]}$ of linear functions

$$\mathbf{j}_I(A) = I a_2 + (1 - I) a_1, \quad I \in [0,1] \quad (2.1)$$

where the *degree of risk* is a constant value I ,

- the family $\{\mathbf{j}_r\}_{r \in \Gamma}$ of non linear functions

$$\mathbf{j}_r(a_1, a_2) = a_1 + \mathbf{r}(a_1, a_2)(a_2 - a_1) \quad (2.2)$$

where the *degree of risk* is not constant, but a function $\mathbf{r} \in \Gamma$ and Γ is the set of the $C^{(1)}$ functions $\mathbf{r}(a_1, a_2) : D \rightarrow [0,1]$, $D = \{(a_1, a_2) : a_2 > a_1 > 0\}$, such that :

- $\mathbf{r}(a_1, a_2)$ is (strictly) decreasing in the first variable
- $\mathbf{r}(a_1, a_2)$ is (strictly) increasing in the second variable

- c) $\mathbf{r}(a_1, a_2) \rightarrow 0$ as $a_1 \rightarrow a_2$
d) $\mathbf{j}_r(a_1, a_2)$ is increasing in a_1 .

In this case the *degree of risk* depends on (a_1, a_2) and precisely on the position on the real axis of the interval and on its width.

We may remark that conditions a) and b) force $\mathbf{r}(a_1, a_2)$ to be valued in the open interval $]0,1[$. Furthermore the condition a) and the regularity of $\mathbf{r}(a_1, a_2)$ imply the strict increasing monotonicity of $\mathbf{j}_r(a_1, a_2)$ in a_2 .

It is obvious that the *evaluation function* defined in (2.2) is a generalization of the one in (2.1). In fact if we consider $\mathbf{r}(a_1, a_2)$ constant, we have

$$\mathbf{j}_r(a_1, a_2) = a_1 + \mathbf{h}(a_2 - a_1) = \mathbf{h}a_2 + (1 - \mathbf{h})a_1 = \mathbf{j}_l(a_1, a_2) \text{ with } \mathbf{l} = \mathbf{h}$$

Using the evaluation function defined in (2.1) we may provide the following order relation:

Def.2.2. Given $A = [a_1, a_2]$ and $B = [b_1, b_2]$ in I , and the *evaluation function* $\mathbf{j}_l(\cdot)$, we state that B is *preferred* to A , in symbols

$$A \prec_{j_l} B \text{ if and only if } \mathbf{j}_l(A) < \mathbf{j}_l(B).$$

and that A is equivalent to B

$$A \approx_{j_l} B \text{ if and only if } \mathbf{j}_l(A) = \mathbf{j}_l(B)$$

Obviously, this order relation on I generates equivalence classes with infinite elements; in fact the interval $A = [a_1, a_2]$ is equivalent to the intervals $[a_1 + \mathbf{d}, a_2 + \mathbf{g}]$ with $\mathbf{d} = \mathbf{g} - \frac{\mathbf{g}}{\mathbf{l}}$.

Using the evaluation function defined in (2.2) we have a slightly different order relation:

Def 2.3. Given $A = [a_1, a_2]$ e $B = [b_1, b_2]$ in I , and an *evaluation function* \mathbf{j}_r , we state that

$$A \prec_{j_r} B \text{ if and only if } \mathbf{j}_r(A) < \mathbf{j}_r(B), \text{ or } \mathbf{j}_r(A) = \mathbf{j}_r(B) \text{ and } \mathbf{r}(a_1, a_2) > \mathbf{r}(b_1, b_2).$$

and we say that

$$A \approx_{j_r} B \text{ if and only if } \mathbf{j}_r(A) = \mathbf{j}_r(B) \text{ and } \mathbf{r}(a_1, a_2) = \mathbf{r}(b_1, b_2)$$

This type of ranking function has been introduced for right fuzzy numbers² in [10].

The following theorem establishes a very strong result concerning the equivalence classes on the ordered space (I, \prec_{j_r})

Theorem 2.1. Let \mathbf{j}_r be a not constant ranking function on I . Then the equivalence classes of intervals generated by \mathbf{j}_r are singletons [10].

It is easy to show that, if $\mathbf{r}(a_1, a_2)$ is constant, the theorem is not valid, as the equivalence classes of intervals generated by $\mathbf{j}_r(\cdot)$ are the same described for $\mathbf{j}_l(\cdot)$.

Now we extend the previous ranking functions to fuzzy numbers.

Let F be the set of *fuzzy numbers* on the universe R ; remember that $\tilde{A} \in F$ can be defined by its \mathbf{a} -cuts $A^{\mathbf{a}}$, $A^{\mathbf{a}} = [a_1^{\mathbf{a}}, a_2^{\mathbf{a}}]$, $\mathbf{a} \in [0,1]$.

Def. 2.4. Given the *evaluation function* $\mathbf{j}(\cdot)$ defined on the \mathbf{a} -cuts of \tilde{A} by

$$\mathbf{j}_l(A^{\mathbf{a}}) = \mathbf{l}a_2^{\mathbf{a}} + (1 - \mathbf{l})a_1^{\mathbf{a}} \quad \mathbf{a} \in [0,1]$$

we define on the set of fuzzy number F the following *evaluation function*

² A right fuzzy number is a fuzzy number for which the triplet is $(a_1 = a_2, a_3)$

$$\Phi_I(\tilde{A}) = 2 \int_0^1 \mathbf{aj}_I(A^a) da \quad (2.3)$$

Def. 2.5. Given the *evaluation function* $\mathbf{j}(\cdot)$ defined on the \mathbf{a} -cuts of \tilde{A} by

$$\mathbf{j}_r(A^a) = a_1^a + \mathbf{r}(a_1^a, a_2^a)(a_2^a - a_1^a) \quad \mathbf{a} \in [0,1]$$

we define on the set of fuzzy number F the following *evaluation functions* $\Phi_r : F \rightarrow R$

$$\Phi_r(\tilde{A}) = 2 \int_0^1 \mathbf{aj}_r(A^a) da \quad (2.4)$$

These functions are average values of the fuzzy number made by a particular normalized Stieltjes measure on $[0,1]$. In fact if we define average value of A the value

$$\Phi_S^j(\tilde{A}) = \int_0^1 \mathbf{j}(A^a) dS(\mathbf{a}) \quad (2.5)$$

where \mathbf{j} is a generic evaluation function on real intervals and S is an additive measure on $[0,1]$, we obtain (2.3) if $\mathbf{j} = \mathbf{j}_I$ and (2.4) if $\mathbf{j} = \mathbf{j}_r$, and S the Stieltjes measure generated by the function $s(\mathbf{a}) = \mathbf{a}^r$ with $r=2$. All the results are still valid $\forall r > 0$ but we prefer $r=2$ because in this case S gives more weight to the high values of $\mathbf{a} \in [0,1]$. It easy to verify that particular choices of \mathbf{j}_I and S let (2.5) coincide with other comparison indexes (Adamo [1], Tsumura [16], Yager [19], Campos-Gonzales [4],[5]). In particular if $\mathbf{j} = \mathbf{j}_I$ and $\tilde{A} = (a_1, a_3, a_2)$, (2.5) turns into the convex combination between the optimistic and pessimistic choice introduced in [9].

Now, let \tilde{A}, \tilde{B} be in F and the *evaluation function* $\Phi_I(\cdot)$ be defined as above.

Def.2.6. We state that \tilde{B} is Φ_I -preferred to \tilde{A} , in symbols

$$\tilde{A} \prec_{\Phi_I} \tilde{B} \quad \text{if and only if} \quad \Phi_I(\tilde{A}) < \Phi_I(\tilde{B}) \quad (2.6)$$

This is a crisp preorder on R and an order relation on the quotient set generated by the equivalence relation

$$\tilde{A} \approx_{\Phi_I} \tilde{B} \quad \text{if and only if} \quad \Phi_I(\tilde{A}) = \Phi_I(\tilde{B})$$

Using the definition 2.2, the (2.6) is equivalent to

$$\tilde{A} \prec_{\Phi_I} \tilde{B} \quad \text{if and only if} \quad \mathbf{j}_I[\Phi_0(\tilde{A}), \Phi_1(\tilde{A})] < \mathbf{j}_I[\Phi_0(\tilde{B}), \Phi_1(\tilde{B})]$$

3 Preliminaries

As we are interested in maximization problem, it is natural to introduce some concavity hypotheses and recall some classical results about this topic.

Property 3.0. If g is a one variable differentiable function on the convex K , then

$$g \text{ is concave if and only if } g(x) \leq g(y) + g'(y)(y-x), \quad \forall x, y \in K \quad (3.1)$$

Theorem 3.1. [12] Let K be a convex set in R^n , $\bar{x} \in K$ and $h: R^n \rightarrow R$ be differentiable in \bar{x} . If \bar{x} is a local maximum then

$$\langle \nabla h(\bar{x}), x - \bar{x} \rangle \leq 0 \quad \forall x \in K.$$

Theorem 3.2. [12] Let K be a convex set in R^n , $h:R^n \rightarrow R$ be a concave function and $\bar{x} \in K$. Then the condition “there is a subgradient $\partial h(\bar{x})$ such that for all $x \in K$ $\langle \partial h(\bar{x}), x - \bar{x} \rangle \leq 0$ “ it is necessary and sufficient in order that \bar{x} is a global maximum of $h(x)$ in K .

Lemma 3.3. Let K be a convex subset of R and $g:[0,1] \times K \rightarrow R$, $u=g(t,x)$, $g \in C^{(2)}$, a function which is concave in x ; then, if we define

$$G(x) = \int_0^1 t g(t,x) dt, \text{ also } G(x) \text{ is concave.}$$

The proof follows immediately, noting that by Property 3.0., as g is concave in x , $\forall t \in [0,1]$ it results

$$g(t,x) \leq g(t,y) + \frac{\partial g}{\partial y}(t,y)(y-x) \quad \forall x,y.$$

Therefore, by using the monotonicity and the linearity of the integral, we obtain

$$\begin{aligned} G(x) &= \int_0^1 t g(t,x) dt \leq \int_0^1 t g(t,y) dt + (y-x) \int_0^1 t \frac{\partial g}{\partial y}(t,y) dt = \\ &= \int_0^1 t g(t,y) dt + (y-x) \frac{\partial}{\partial y} \int_0^1 t g(t,y) dt = G(y) + G'(y)(y-x) \end{aligned}$$

which gives the required concavity condition.

Consider now a parameterized optimization problem of the form

$$\max_{t \in R} f(t,q), \quad q \in R \quad (3.2.)$$

(or similarly for minimum) where f is a regular function. Assume that $t(q)$ be a single-valued maximization choice of t and denote $v(q) = f(t(q),q)$ the value attained by f at the solution $t(q)$ to problem (3.2.).

Theorem 3.4. (Envelope Theorem) ([15], pag 327) If $f \in C^{(1)}$, then $t(q)$ and $v(q)$ are differentiable and it results

$$\frac{dv(\bar{q})}{dq} = \frac{\partial f(t(\bar{q}),\bar{q})}{\partial q} \quad (3.3)$$

That is, the fact that $t(q)$ is determined by maximizing the function $f(.,q)$ has the implication that in computing the first order effects of changes in q on the maximum value, we can equally well assume that the maximizer will not adjust. The only effect of any consequence is the direct effect.

4 The optimisation problem.

Given a fuzzy number $\tilde{A} \in F$ and the function $f(a,x):R^2 \rightarrow R$, we consider the following fuzzy extension induced by f . Let A^a be the \mathbf{a} -cuts of \tilde{A} : then we define $\tilde{f}:F \times R \rightarrow F$, $\tilde{f}(\tilde{A},x) = \tilde{Y}$ where \tilde{Y} can be defined through its \mathbf{a} -cuts Y^a , as follows

$$Y^a(x) = \tilde{f}(A^a,x) = [z_1^a(x), z_2^a(x)] \quad (4.1)$$

and $z_1^a(x)$, $z_2^a(x)$ are defined, $\forall \mathbf{a} \in [0,1]$,

$$z_1^a(x) = \min_{a_1^a \leq a \leq a_2^a} f(a, x) \quad ; \quad z_2^a(x) = \max_{a_1^a \leq a \leq a_2^a} f(a, x). \quad (4.2)$$

In the following, we consider $f \in C^{(2)}$; moreover, $\forall \mathbf{a} \in [0,1]$ we call $a_L^a(x)$ and $a_U^a(x)$ respectively the minimizing and maximizing functions, that is

$$a_L^a(x) = \arg \min_{a_1^a \leq a \leq a_2^a} f(a_L^a(x), x)$$

$$a_U^a(x) = \arg \max_{a_1^a \leq a \leq a_2^a} f(a_U^a(x), x)$$

hence

$$f(a_L^a(x), x) = z_1^a(a, x) \text{ and } f(a_U^a(x), x) = z_2^a(a, x)$$

and we suppose from now that $a_L^a(x)$ and $a_U^a(x)$ are single-valued and consequently continuous [16].

The *evaluation function*, defined in (2.3), is now the following function of the real variable x of R :

$$\forall x \in R \quad \Phi_I(\tilde{Y}(x)) = 2 \int_0^1 \mathbf{a} \mathbf{j}_I(\tilde{f}(A^a, x)) d\mathbf{a} \quad (4.3)$$

Therefore, the optimisation problem we wish to solve is:

$$\max_{x \in K} \Phi_I(\tilde{Y}(x)) = \max_{x \in K} 2 \int_0^1 \mathbf{a} \mathbf{j}_I(\tilde{f}(A^a, x)) d\mathbf{a} \quad (4.4)$$

where K is a convex subset of R .

In the following we will write $\Phi_I(x) = \frac{1}{2} \Phi_I(\tilde{Y}(x))$.

Theorem 4.1. Let $f(a, x): R^2 \rightarrow R$, $f \in C^{(2)}$, and $\tilde{f}(A^a, x): F \times R \rightarrow F$, be the fuzzy extension of f . If:

i) f is convex in a and concave in x ,

ii) $\forall \mathbf{a} \in [0,1]$, $a_L^a(x) \in]a_1^a, a_2^a[$

then \bar{x} is the maximum for f on K , with respect to the evaluation \mathbf{j}_I , if and only if

$$(x - \bar{x}) \cdot \int_0^1 \mathbf{a} \left[\mathbf{I} \frac{\partial f}{\partial x}(a_U^a(\bar{x}), \bar{x}) + (1 - \mathbf{I}) \frac{\partial f}{\partial x}(a_L^a(\bar{x}), \bar{x}) \right] d\mathbf{a} \leq 0 \quad \forall x \in K \quad (4.5)$$

Proof. First of all, we recall that, as we supposed above, $a_L^a(x)$ and $a_U^a(x)$ are single-valued: moreover, since $f \in C^{(2)}$, the differentiability of $a_L^a(x)$ and $a_U^a(x)$ follows from the regularity of f .

Let us begin proving the concavity with respect to x of the function $\mathbf{j}_I(\tilde{f}(A^a, x)) = \mathbf{j}_I(\mathbf{a}, x)$ which appears in (4.4). As $\mathbf{j}_I(\mathbf{a}, x) \in C^{(2)}$, this is equivalent to show that

$$\frac{\partial^2 \mathbf{j}_I(\mathbf{a}, x)}{\partial x^2} \leq 0 \quad (4.6)$$

Now, the integral argument in (4.4), has the expression

$$\mathbf{j}_I(\tilde{f}(A^a, x)) = \mathbf{j}_I(\mathbf{a}, x) = \mathbf{I} z_2^a(x) + (1 - \mathbf{I}) z_1^a(x)$$

where z_1^a and z_2^a are defined as in (4.2).

Firstly we compute $\forall \mathbf{a} \in [0,1]$

$$\frac{\partial \mathbf{j}_I(\mathbf{a}, x)}{\partial x} = \mathbf{I} \frac{dz_2^a}{dx}(x) + (1 - \mathbf{I}) \frac{dz_1^a}{dx}(x)$$

where $\frac{dz_1^a}{dx}(x) = \frac{\partial f}{\partial a}(a_L^a(x), x) \frac{da_L^a}{dx}(x) + \frac{\partial f}{\partial x}(a_L^a(x), x)$ and similarly for $\frac{dz_2^a}{dx}(x)$.

Observe that from the convexity of f with respect to a , and from the observation that $a_U^a(x)$ is single-valued and continuous, the maximizing function $a_U^a(x)$ lies on the boundary of $[a_1^a, a_2^a]$, and $\forall \mathbf{a} \in [0,1]$ its value is constant with respect to x , it is equal either to a_1^a or to a_2^a and consequently we have

$$\frac{da_U^a}{dx} = 0 \quad (4.7).$$

Moreover, being $a_L^a(x) \in]a_1^a, a_2^a[$, we can apply the *envelope theorem* to $z_1^a(x) = \min_{a_1^a \leq a \leq a_2^a} f(a, x)$,

obtaining

$$\frac{\partial f}{\partial a}(a_L^a(x), x) = 0 \quad (4.8).$$

Thus we can conclude

$$\begin{aligned} \frac{dz_1^a}{dx}(x) &= \frac{\partial f}{\partial a}(a_L^a(x), x) \frac{da_L^a}{dx}(x) + \frac{\partial f}{\partial x}(a_L^a(x), x) = \frac{\partial f}{\partial x}(a_L^a(x), x) \\ \frac{dz_2^a}{dx}(x) &= \frac{\partial f}{\partial a}(a_U^a(x), x) \frac{da_U^a}{dx}(x) + \frac{\partial f}{\partial x}(a_U^a(x), x) = \frac{\partial f}{\partial x}(a_U^a(x), x) \end{aligned}$$

Therefore

$$\frac{\partial \mathbf{j}_I(\mathbf{a}, x)}{\partial x} = \mathbf{I} \frac{\partial f}{\partial x}(a_U^a(x), x) + (1 - \mathbf{I}) \frac{\partial f}{\partial x}(a_L^a(x), x) \quad (4.9)$$

and consequently the concavity inequality for \mathbf{j}_I can be written as follows

$$\begin{aligned} \frac{\partial^2 \mathbf{j}_I(\mathbf{a}, x)}{\partial x^2} &= \frac{\partial}{\partial x} \frac{\partial \mathbf{j}_I(\mathbf{a}, x)}{\partial x} = (1 - \mathbf{I}) \left\{ \frac{\partial^2 f}{\partial x \partial a}(a_L^a(x), x) \frac{da_L^a(x)}{dx} + \frac{\partial^2 f}{\partial x^2}(a_L^a(x), x) \right\} + \\ &+ \mathbf{I} \left\{ \frac{\partial^2 f}{\partial x \partial a}(a_U^a(x), x) \frac{da_U^a(x)}{dx} + \frac{\partial^2 f}{\partial x^2}(a_U^a(x), x) \right\} \leq 0 \quad (4.10) \end{aligned}$$

Now we can observe that by (4.7) the term $\frac{\partial^2 f}{\partial x \partial a}(a_U^a(x), x) \frac{da_U^a(x)}{dx}$ is null and moreover that the

derivative $\frac{\partial^2 f}{\partial a^2}(a_L^a(x), x)$ is non negative, as f is convex in a .

As a consequence of the *envelope theorem*, we can compute the effects of changing x on $a_L^a(x)$ by differentiating the first-order condition: in fact we obtain

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial a}(a_L^a(x), x) \right) = 0$$

whence

$$\frac{\partial^2 f}{\partial a^2}(a_L^a(x), x) \frac{\partial a_L^a(x)}{\partial x} + \frac{\partial^2 f}{\partial a \partial x}(a_L^a(x), x) = 0 \quad (4.11)$$

If $\frac{\partial^2 f}{\partial a^2}(a_L^a(x), x) = 0$, then from (4.11) we infer $\frac{\partial^2 f}{\partial a \partial x}(a_L^a(x), x) = 0$ and the condition (4.10) becomes

$$\frac{\partial^2 \mathbf{j}_1(\mathbf{a}, x)}{\partial x^2} = \mathbf{I} \frac{\partial^2 f}{\partial x^2}(a_U^a(x), x) + (1 - \mathbf{I}) \frac{\partial^2 f}{\partial x^2}(a_L^a(x), x) \leq 0$$

which follows immediately from the concavity of f in x .

If $\frac{\partial^2 f}{\partial a^2}(a_L^a(x), x) > 0$, then we obtain from (4.11)

$$\frac{\partial a_L^a}{\partial x} = - \frac{\frac{\partial^2 f(a_L^a(x), x)}{\partial x \partial a}}{\frac{\partial^2 f(a_L^a(x), x)}{\partial a^2}}$$

and therefore

$$\frac{\partial^2 \mathbf{j}_1(f(a, x))}{\partial x^2} = (1 - \mathbf{I}) \left\{ - \frac{\left[\frac{\partial^2 f(a_L^a(x), x)}{\partial x \partial a} \right]^2}{\frac{\partial^2 f(a_L^a(x), x)}{\partial a^2}} + \frac{\partial^2 f}{\partial x^2}(a_L^a(x), x) \right\} + \mathbf{I} \left\{ \frac{\partial^2 f}{\partial x^2}(a_U^a(x), x) \right\}$$

Since the hypothesis i) guarantees that $\frac{\partial^2 f}{\partial x^2}(a, x) \leq 0, \forall a$ and the first term in the first quote is also negative, as we are in the case $\frac{\partial^2 f}{\partial a^2}(a_L^a(x), x) > 0$, finally we proved that

$$\frac{\partial^2 \mathbf{j}_1(f(a, x))}{\partial x^2} \leq 0$$

which gives us the concavity of $\mathbf{j}_1(\mathbf{a}, x)$.

Using the result of Lemma 3.3, we can also argue the concavity of $\Phi_1(x)$. Moreover, since $\Phi_1(x)$ is concave, Lemma 3.2. ensures that the inequality

$$\Phi_1'(\bar{x})(x - \bar{x}) \leq 0 \quad \forall x \in K \quad (4.12)$$

is a necessary and sufficient condition that \bar{x} be the maximum for $\Phi_1(x)$ in K .

Finally, if we remember the definition of $\Phi_1(x)$ given in (4.3), and the equality (4.9), we can write (4.12) in an equivalent way as follows:

$$\int_0^1 \mathbf{a} \frac{\partial \mathbf{j}_I(\mathbf{a}, \bar{x})}{\partial x} \cdot (x - \bar{x}) d\mathbf{a} =$$

$$= \int_0^1 \mathbf{a} \left[\mathbf{I} \frac{\partial f}{\partial x}(a_U^{\mathbf{a}}(\bar{x}), \bar{x}) + (1 - \mathbf{I}) \frac{\partial f}{\partial x}(a_L^{\mathbf{a}}(\bar{x}), \bar{x}) \right] \cdot (x - \bar{x}) d\mathbf{a} \leq 0 \quad \forall x \in K$$

which gives the thesis.

5 An example

If we consider the following simple function

$$f(a, x) = \tilde{a}^2 x - x^2, \quad a \in [1, 3]$$

for $\mathbf{a} \in [0, 1]$ we have

$$a_1^{\mathbf{a}} = \mathbf{a} + 1, \quad a_3^{\mathbf{a}} = 3 - \mathbf{a}$$

and then

$$y^{\mathbf{a}} = f(a^{\mathbf{a}}, x) = [a_1^{\mathbf{a}}, a_3^{\mathbf{a}}] x - x^2 = [(\mathbf{a} + 1), (3 - \mathbf{a})] x - x^2 = [(\mathbf{a} + 1)^2 x, (3 - \mathbf{a})^2 x] - x^2 =$$

$$= [(\mathbf{a} + 1)^2 x - x^2, (3 - \mathbf{a})^2 x - x^2]$$

as $\mathbf{I} \in [0, 1]$, the problem becomes:

$$2 \int_0^1 \frac{\partial \Phi}{\partial x}(\mathbf{a}, x) d\mathbf{a} = 2 \int_0^1 \mathbf{a} [-4x + [(1 - \mathbf{I})(\mathbf{a} + 1)^2 + \mathbf{I}(3 - \mathbf{a})^2]] d\mathbf{a} = 0$$

$$2 \int_0^1 \frac{\partial \Phi}{\partial x}(\mathbf{a}, x) d\mathbf{a} = 2 [-2x\mathbf{a}^2]_0^1 + 2(1 - \mathbf{I})x \left[\frac{\mathbf{a}^4}{4} + 2\frac{\mathbf{a}}{3} + \frac{\mathbf{a}^2}{2} \right]_0^1 + 2\mathbf{I} \left[9\frac{\mathbf{a}^2}{2} + \frac{\mathbf{a}^4}{4} - 2\mathbf{a}^3 \right]_0^1 =$$

$$= \frac{11}{12}x + \frac{5}{3}\mathbf{I} = 0$$

which has the unic solution:

$$\bar{x} = -\frac{20\mathbf{I}}{11}$$

Conclusions

In this paper we have introduced two different evaluation functions for intervals and by the first of them we have defined a ranking on fuzzy numbers by average value. The optimisation problem has been treated only for the first case as the evaluation of (4,6) using the second case, seems at the moment really complicated. In any case we think that the proposed result let the possibility to treat the fuzzy optimisation problem without having any information about the minimum and the maximum of the function f. This opportunity should give the possibility to build an algorithm to reach the solution easier. We are already working on the second type of ranking to reach a necessary and sufficient condition like that in theorem (4,1). Another interesting research in which we are involved is the same optimisation problem in the average value of the fuzzy number is obtained using non-additive measures like a fuzzy measure, using Choquet's integral.

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