\\ 368 \\

Implied Trees in Illiquid Markets: a Choquet Pricing Approach

by

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July 2001

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IMPLIED TREES IN ILLIQUID MARKETS: A CHOQUET PRICING APPROACH

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ABSTRACT

Implied trees are necessary to implement the risk neutral valuation approach and standard methodologies for their derivation are based on the validity of the Put Call Parity. However, in illiquid markets the Put Call Parity fails to hold and the uniqueness of the artificial probabilities leaves room to an interval. The contribution of this paper is twofold. First we propose a methodology for the derivation of implied trees in illiquid markets. Such a methodology, by contrast with standard ones, takes into account the information stemming both from call and put prices. Secondly, we set up a framework for pricing derivatives written on an underlying traded on an illiquid market. To this end we have extended the Choquet integral definition to account for interval payoffs of the underlying. The price interval we obtain may be interpreted as a bid-ask price quoted by the intermediary that has issued the derivative security.

Keywords: Binomial Model, Put-Call Parity, Choquet Pricing.
JEL classification: G13, G14.

1. INTRODUCTION

The main drawback of the Black and Scholes equation and of the Cox-Ross-Rubinstein binomial implementation is that the stock evolves along a risk neutral binomial tree with constant volatility. Derman and Kani (1994) suggested a preference free model that is consistent with the smile effect and the term structure of implied volatility. The model derives an implied binomial tree using call options for the upper part and put options for the lower part of the tree, implicitly assuming that the put call parity (PCP) holds. However, in illiquid markets, where the PCP is not fulfilled, the implied tree based on call prices is different from the one obtained using put prices.

The aim of this paper is twofold: to propose a methodology for the derivation of implied trees in illiquid markets and to set up a framework for pricing derivatives written on the underlying asset.

[1] The authors acknowledge financial support from MURST. The authors thank Bernard De Baets for helpful comments and suggestions on a previous version of the present paper. Usual disclaimer applies.

Violations to PCP have been detected for American options on US stocks and stock indices in many empirical works (e.g. Stoll (1969), Gould and Galai (1974), Klemowski and Resnick (1979), Evnine and Rudd (1985)). In order to overcome the problems linked with the estimation of the early exercise premium, more recent studies tested the PCP using European options: among others, Kamara and Miller (1995), Chesney, Gibson and Loubergé (1995), Mittnick and Rieken (2000), Cavallo and Mammola (2000), report weaker, but still significant violations. In particular, Kamara and Miller (1995) found that violations to PCP are not due to market inefficiency and reflect premia for liquidity risk (Kamara and Miller (1995)). They show that, as liquidity risk increases, both the size and the frequency of deviations from PCP increase. Moreover, liquidity risk widens bid-ask spreads in option prices and can lead to overpricing or underpricing of call and put options relative to their PCP implied value. When the Black and Scholes formula is used, a difference in the implied volatility of call and put prices is normally observed. In particular, the implied volatility computed from put prices is generally bigger then the implied volatility computed from call prices (see e.g. Chesney, Gibson and Loubergé (1995) and Cavallo and Mammola (2000)). The effect of pricing calls and puts with the same volatility estimate yields in general to mispricing phenomena. This leads to the conclusion that, in illiquid markets, call and put prices focusing on different aspects of the underlying process, carry different information on the volatility of the latter. By taking into account only call (put) prices and relying on PCP results in a loss of information about the underlying process. Therefore standard methodologies used to derive implied trees, e.g. Derman and Kani (1994), when used in a illiquid market suffer from this shortfall.

In this paper we extend the Derman and Kani model to take into account liquidity risk, thus allowing for violations to the PCP. We develop a tree that incorporates and reflects liquidity risk, the smile effect and the time to expiration bias of volatility for call and put options. This tree is suitable to value any derivative written on the underlying asset. When the PCP fails to hold, the uniqueness of the artificial probabilities leaves room to an interval. In order to bound the artificial probabilities and the underlying stock prices at each node of the tree, we derive two implied trees: one using only call options and the other one using only put options. In this way we are able to bound the artificial probabilities and the underlying stock prices at each node of the tree, generating an implied tree which is consistent with both call and put prices. From now on we denote this tree as the PC-implied tree. In order to be able to take expectations on such a binomial tree, we extend the Choquet integral definition to take into account interval values for the underlying asset instead of point estimates. The price interval for the derivative security may be interpreted as a bid ask price quoted by the intermediary.

The plan of the paper is the following. In section 2 we briefly illustrate the effect of PCP violations on the Derman and Kani model. In section 3 we formally set up the algorithm for the derivation of the PC-implied tree. In section 4 we apply and extend the Choquet integral definition to take into account interval values of the derivative payoff. In section 5 we explain the pricing process of a derivative written on the underlying asset. The last section concludes.

2. THE DERMAN AND KANI MODEL AND PCP VIOLATIONS

Derman and Kani (1994) develop a preference free model based on a no-arbitrage argument, for the derivation of the implied tree. The tree can be used to value options from easily observable data. They extend the Black and Scholes pricing framework, requiring the volatility to depend both on time to expiration and on the value of the underlying asset. They do not assume a particular parametric form for the volatility, but they require the option prices to fit the empirically estimated smile curve. The standard binomial model is thus replaced by a distorted or implied tree, with different volatility at each node.

The implied binomial tree, consistent with the smile effect and with the term structure of implied volatilities, is built by forward induction starting from the first node. Let r be the risk free rate, S(0) the initial stock price and C(S(0),1) the price of an at the money (ATM) call, i.e. with strike price equal to S(0), expiring at T=1. By solving equations (1) (2) and (3):

$$F = S(0)(1+r) = \pi S(H) + (1-\pi)S(L) \quad (1)$$

$$C(S(0),1) = 1/(1+r)\,\pi\,[S(H) - S(0)] \quad (2)$$

$$S(H)\,S(L) = S(0)^2 \quad (3)$$

the model provides three outputs: the risk neutral probability of an up move, π, and the value of the stock at time one in state up, S(H), and in state down, S(L). Figure 1 depicts the first level of the tree. Equations (1) and (2) require respectively the stock price and the call price to be consistent with the risk neutral valuation approach with probability π (note that as the call is in the money, it pays out only in the upper state); equation (3) is a centring condition analogous to the one introduced by Cox, Ross and Rubinstein (1979) that essentially require the tree to develop around the current spot price of the underlying.

In general, at time n there are 2n +1 unknowns i.e. n+1 stock values and n risk neutral probabilities.

To determine the unknowns Derman and Kani use n equations representing the risk neutral valuation of the forward price of the stock, n equations representing the risk neutral valuation of call (put) prices for the upper (lower) half of the tree, with strikes equal to the stock prices at the

previous level and expiration at time *n* and fill the remaining degree of freedom by means of the centring condition. The procedure is iterated up to the last expiration date.

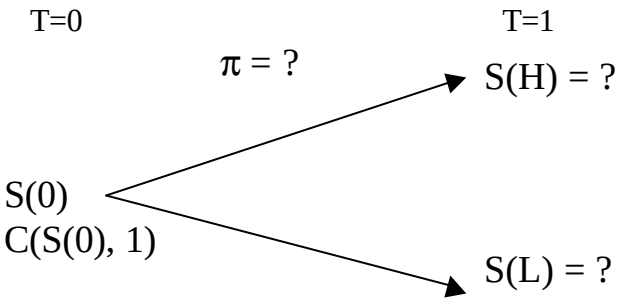

Figure 1. The first level of the Derman and Kani tree.

In such a way, Derman & Kani construct a tree which is consistent with the smile since they use call and put values interpolated from the smile curve[2]. Using call prices for the nodes in the upper half and put prices for the nodes in the lower half of the tree has the computational advantage that, at each step, there are only a few nodes in which the option is in the money.

It has to be stressed that this methodology heavily relies on the validity of the PCP. In fact, if the PCP holds, the equivalence between a call and a put makes it indifferent to use either a call or a put price and thus justifies the use of the most convenient one. However, if the PCP is violated, as it is the case in illiquid markets, the indifference between the use of a call and a put price is lost and the implied tree becomes sensitive to the type of option used to derive it.

In sum, when the PCP fails to hold the Derman and Kani methodology may lead to a tree that ignores the different information carried by in the money call prices (in the lower part of the tree) and by in the money put prices (in the upper part).

To better illustrate this point we compute for example the underlying asset price at time one in state up. The solution of equations (1) to (3), i.e. the use of a call to derive the implied tree, yields the following price:

$$S(H) = \frac{S(0) * C(S(0),1) + \frac{S(0)^2}{1+r}}{S(0) - C(S(0),1)} \qquad (9)$$

If a put P(S(0), 1) were used, i.e. if equation (2) were replaced by the following:

$$P\,(K=S(0),1) = 1/(1+r)\,(1-\pi)\,[S(0) - S(L)] \qquad (2')$$

[2] The smile curve is obtained using call prices for strikes above the underlying spot and put prices for strikes below.

The solution of equations (1), (2') and (3) would yield the following price for the spot at time one in state up:

$$S(H) = \frac{S(0)^2 + P(S(0),1) * S(0)}{\frac{S(0)}{1+r} - P(S(0),1)} \tag{10}$$

It is easy to prove that if PCP holds, equations (9) and (10) are equivalent, if it does not hold, different prices for the underlying asset are in general obtained. In fact, if the call (put) is worth less then its replicating portfolio, i.e. $C(S(0))<P(S(0))+S(0)-X(1+r)$, the corresponding put (call) is worth more. It follows that the underpriced call (put) implies a price for S(H) lower and a price for S(L) higher then the corresponding overpriced put (call) i.e., the volatility implied by underpriced calls or puts is lower then the one implied by overpriced calls or puts. As a consequence, in markets where the PCP is not fulfilled, different trees for the same underlying asset may be implied depending on the type of option price used. Therefore in markets where PCP violations are observed, in order to apply the Derman and Kani methodology, the problem of an arbitrary choice of the type of option (call vs. put) to be used at each node needs to be solved.

3. THE PC-IMPLIED TREE

In this section we set up a methodology for the derivation of an implied tree that can be used in illiquid markets and is consistent with the whole information available in the market. To this end, we build an implied tree taking at each node stock values and artificial probabilities implied by both call and put prices. Our method basically extends Derman and Kani's in order to use call prices also in the lower part of the tree and put prices also in the upper part. As the PCP is not fulfilled, using an additional set of n equations for in the money call and put prices would make the system impossible.

We propose to develop two implied trees, one using only calls and one using only puts and to aggregate the conflicting information by taking the call and put implied stock prices and probabilities as bounds for an interval of prices and probabilities respectively. Such a tree with interval values for the underlying stock prices and probabilities will be denoted as PC-implied tree.

Given the stylised empirical fact that volatility varies across both moneyness and time to expiration, there are basically two ways of constructing an implied tree, depending on which phenomenon we want to capture. In modelling dependence on time to expiration, we are faced with the problem of a limited number of expiration dates for the options traded in the market. Therefore, if we need an implied tree consistent with both the empirical biases, we are constrained to build a tree with only a few levels and we can read the relevant call or put price directly from the data set by interpolation.

We do not examine in detail this case. By contrast, if we accept to build a tree that is consistent only with the first bias, we can suppose that the smile is the same across dates and we can build a tree with as many levels as needed.

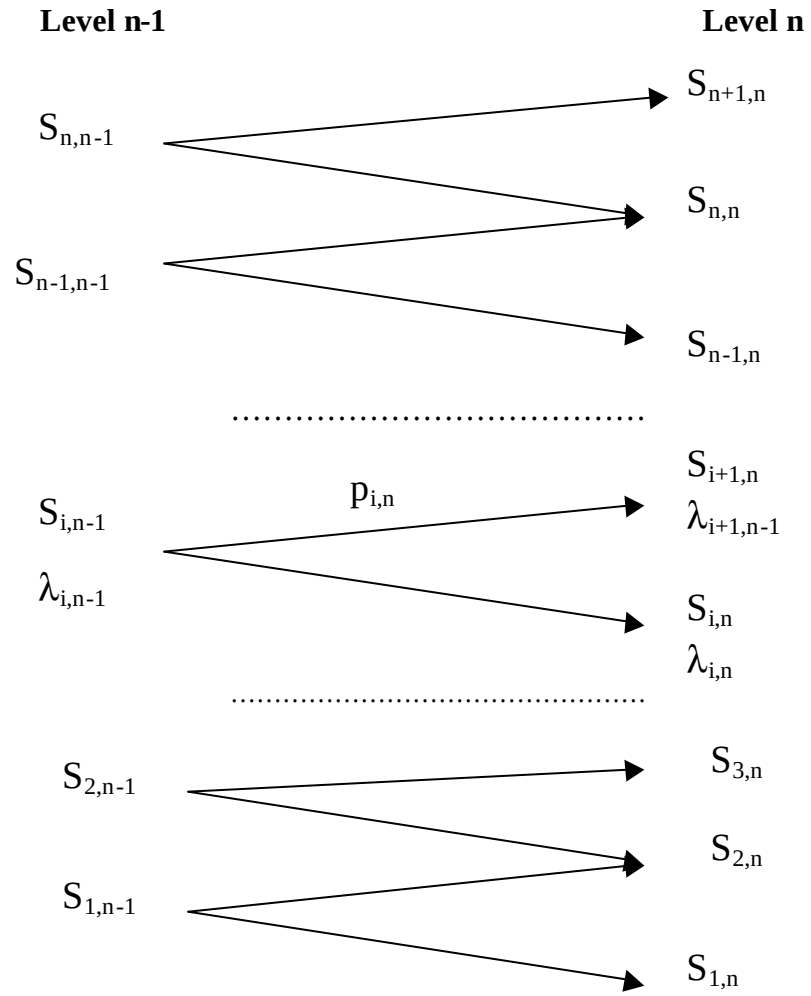

Figure 3. Levels n-1 and n of the tree.

Let j=0,…,n be the number of levels of the tree. As we assume that the tree recombines, let i=1,...,j+1 be the number of nodes at level j. We use forward induction to compute level n variables from the inputs of level n-1. We compute first the implied tree using only call prices, then the implied tree using only put prices and as a last step we take the implied prices and probabilities from the previously computed trees as bounds for a unique PC-implied tree.

We first show how to compute the call implied tree. The initial inputs are the riskless interest rate, the stock price at time zero and the smile function for call prices. We read from the smile function the volatility corresponding to the strike price needed in order to have an at the money call. We generate a binomial tree with constant volatility and we compute the price of the call at time zero. We use the stock price, the call price and the centring condition to imply the first level.

In general at level n there are 2n+1 unknowns: n+1 stock prices and n risk neutral probabilities. Figure 3 focuses on levels n-1 and n. The inputs are the n stock prices at level n-1 and the prices of n ATM calls with strikes equal to the n stock prices (computed as explained for the first level). The stock price $S_{i,n-1}$ takes value $S_{i+1,n}$ in state up, and $S_{i,n}$, in state down and the risk neutral probability of an up jump is $p_{i,n}$. The Arrow-Debreu price in node (i,n-1) is $\lambda_{i,n-1}$ and is computed by forward induction as the sum over all paths leading to node (i,n-1) of the product of the transition probabilities discounted at the riskfree rate at each node in each path. All $\lambda_{i,n-1}$ are known since the transition probabilities of levels j=0,...,n-1 have already been implied.

We use the following 2n+1 equations (13) and (14), i.e. we require the theoretical value of n forwards and n call options all expiring at time n to match their market values, and for the remaining equation we use the centring condition. The centring condition, that requires the tree to develop around the spot price of the underlying, is given by equation (11) if the level is even and by equation (12) if the level is odd:

$$S_{(n+1)/2}=S_0 \tag{11}$$

$$S_{(n)/2}S_{(n)/2+1}=S_0^2 \tag{12}$$

The first set of n equations requires the stock price to be consistent with the risk neutral valuation:

$$F_{i,n}=p_{i,n}\,S_{i+1,n}+(1-p_{i,n})S_{i,n} \qquad \forall i=1,..,n \tag{13}$$

where $F_{i,n}=(1+r)\,S_{i,n-1}$ and

$$p_{i,n}=\frac{F_{i,n}-S_{i,n}}{S_{i+1,n}-S_{i,n}} \tag{14}$$

The second set of n equations is given by the risk neutral valuation of n call options, each with strike price equal to $S_{i,n-1}$ for i=1,n, with expiration at time n. The condition imposed on each call option to be at the money grants that only the nodes above the strike contributes to the payoff of the call:

$$C_{i,n-1}=\frac{1}{1+r}\sum_{j=1}^{n}(\lambda_{j,n-1}p_{j,n}+\lambda_{j+1,n-1}(1-p_{j+1,n}))*\max(S_{j+1,n}-S_{i,n-1},0) \tag{15}$$

where to simplify notation $C(S_{i,n-1},n)=C_{i,n-1}$ is the call price at time zero.

In order to separate the effect of the first in the money node from other nodes, we can write equation (15) as follows:

$$(1+r)C_{i,n-1}=\lambda_{i,n-1}p_{i,n}(S_{i+1,n}-S_{i,n-1})+\sum_{j=i+1}^{n}\lambda_{j,n-1}(F_{j,n}-S_{i,n-1}) \tag{16}$$

where the first term depends on the unknown values $p_{i,n}$ and $S_{i+1,n}$ while the second one is computed using quantities that are known at level n. From now on we denote the summation term by Σ.

By solving simultaneously equations (13) and (16) for $S_{i+1,n}$ we get a recursive formula to compute $S_{i+1,n}$ given $S_{i,n}$. For every level j=1,…,n we get:

$$S_{i+1,\,j} = \frac{S_{i,\,j}((1+r)C_{i,\,j-1} - \Sigma) - \lambda_{i,\,j-1} S_{i,\,j-1}[F_{i,\,j} - S_{i,\,j}]}{(1+r)C_{i,\,j-1} - \Sigma - \lambda_{i,\,j-1}[F_{i,\,j} - S_{i,\,j}]} \tag{17}$$

In order to use equation (17) we need an initial node $S_{i,j}$. If the number of nodes is odd we choose the central node to be equal to the current spot. If the number of nodes is even, we substitute equation (12) in equation (17) and we get.

$$S_{(j+3)/2,\,j} = \frac{S_0((1+r)C_0 + \lambda_{(j+1)/2,\,j-1} S_0 - \Sigma)}{\lambda_{(j+1)/2,\,j-1} F_{(j+1)/2,\,j} - (1+r)C_0 + \Sigma} \tag{18}$$

For the nodes below the central values we use the following equation that yields $S_{i,j}$ given $S_{i+1,j}$:

$$S_{i,\,j} = \frac{S_{i+1,\,j}((1+r)C_{i,\,j-1} - \Sigma) - \lambda_{i,\,j-1} F_{i,\,j}[S_{i+1,\,j} - S_{i,\,j-1}]}{(1+r)C_{i,\,j-1} - \Sigma - \lambda_{i,\,j-1}[S_{i+1,\,j} - S_{i,\,j-1}]} \tag{19}$$

By repeating this process at each level we are able to generate the entire tree.

The artificial probabilities of each node must belong to]0,1[. A violation of this condition implies the presence of riskless arbitrage opportunities. Thus, at each iteration, we require the implied stock price in node up to fall above the forward price and the implied stock price in node down to fall below the forward price. If this is not verified, we use the same procedure as Derman and Kani (1994) to override it, i.e. we determine the stock price $S_{i,j}$ by means of the following condition for nodes below the center of the tree:

$$\ln(S_{i,\,j}) + \ln(S_{i+1,\,j}) = \ln(S_{i,\,j-1}) + \ln(S_{i+1,\,j-1})$$

and by means of the following condition for nodes above the center of the tree:

$$\ln(S_{i,\,j}) + \ln(S_{i-1,\,j}) = \ln(S_{i,\,j-1}) + \ln(S_{i-1,\,j-1})$$

Intuitively, for nodes below (above) the center of the tree, we keep the volatility of stock prices in nodes i and i+1 (i-1) at level j the same as in the corresponding nodes at the previous level j-1.

The construction of the put implied tree is analogous to the one already described for the call. The initial inputs are the risk free interest rate, the price of the stock and the price of an ATM put with expiration at time one. The empirically derived smile function for put prices is used to compute the smile-consistent put prices for the next level. Depending on the strike price needed to imply the next node, we read from the smile function the appropriate volatility and we compute the price of a put using a non distorted binomial model with constant volatility.

To imply level n we use the following 2n+1 equations: we require the theoretical risk neutral values of n forwards and n put options expiring at time n to match the interpolated market values, and we

use the centring condition given by equation (11) if the level is even and by equation (12) if the level is odd.

The value of a put with strike $S_{i,n-1}$ is given by the following:

$$P_{i,n-1} = \frac{1}{1+r} \sum_{j=1}^{n} (\lambda_{j,n-1}(1-p_{j,n}) + \lambda_{j-1,n-1}(p_{j-1,n}))*\max(S_{i,n-1} - S_{j,n}, 0) \tag{20}$$

where to simplify notation $P(S_{i,n-1}, n) = P_{i,n-1}$ is the put price today computed using the smile function to obtain the appropriate implied volatility and generating a n step tree with constant appropriate volatility.

Rewriting equation (20) in order to separate the effect of the first in the money node yields:

$$(1+r)P_{i,n-1} = \lambda_{i,n-1}(1-p_{i,n})(S_{i,n-1} - S_{i,n}) + \sum_{j=1}^{i-1} \lambda_{j,n-1}(S_{i,n-1} - F_{j,n}) \tag{21}$$

where the first term depends on the unknown values of $p_{i,n}$, and $S_{i,n}$ while the summation term is computed using quantities that are known at level n, for notational simplicity we denote by Σ' the summation term in equation (21).

Solving equations (13) and (21) for a level j=1,…,n we obtain a recursive formula that provides $S_{i,j}$ given $S_{i+1,j}$:

$$S_{i,j} = \frac{S_{i+1,j}((1+r)P_{i,j-1} - \Sigma') + \lambda_{i,j-1}S_{i,j-1}[F_{i,j} - S_{i+1,j}]}{(1+r)P_{i,j-1} - \Sigma' + \lambda_{i,j-1}[F_{i,j} - S_{i+1,j}]} \tag{22}$$

To use equation (22) we need an initial stock price $S_{i+1,j}$. As in the case of the call, if the number of nodes is odd we take as initial node the central node of the level that is equal to the spot. If the number of nodes is even we substitute equation (12) in equation (22) and we get:

$$S_{(j+1)/2,j} = \frac{S_0(\Sigma' - (1+r)P_0 + \lambda_{(j+1)/2,j-1}S_0)}{\lambda_{(j+1)/2,j-1}F_{(j+1)/2,j} + (1+r)P_0 - \Sigma} \tag{23}$$

For the nodes above the central node we use the following recursive equation that computes $S_{i+1,j}$ given $S_{i,j}$:

$$S_{i+1,j} = \frac{S_{i,j}((1+r)P_{i,j-1} - \Sigma') - \lambda_{i,j-1}F_{i,j}[S_{i,j-1} - S_{i,j}]}{(1+r)P_{i,j-1} - \lambda_{i,j-1}[S_{i,j-1} - S_{i,j}] - \Sigma'} \tag{24}$$

By repeating this process at each level we are able to generate the entire tree.

If an implied stock price does not respect the no arbitrage condition, we use the same procedure explained for call prices to override it. Once we have constructed both the call and the put implied trees, we are able to compute the PC-implied tree.

Let us indicate with $T_{i,j}$ the stock price at node i, level j computed using put prices and with $S_{i,j}$ the stock price at node i, level j computed with call prices, $q_{i,j}$ be the artificial probabilty of an up move

from node (i,j-1) to node (i+1,j) computed using put prices and $p_{i,j}$ be the artificial probability computed using call prices.

For each node, we take $\min(S_{i,j},T_{i,j})$ and $\max(S_{i,j},T_{i,j})$ as the lower and the upper bound respectively for the stock price and $\min(p_{i,j},q_{i,j})$ and $\max(p_{i,j},q_{i,j})$ as the lower and the upper bound respectively for the artificial probability. In this way we are able to incorporate in a unique implied tree all the information stemming from call and put prices.

This is a simple way of deriving interval values for the probabilities and the underlying process from market data. In the following section we explain how to use the PC-tree to take expectations and thus value any derivative on the underlying asset.

4. CHOQUET PRICING WITH INTERVAL VALUES FOR THE UNDERLYING ASSET

The pricing of derivatives by means of the risk-neutral valuation approach implies discounting expected values under the risk-neutral probabilities. When probabilities are crisp values, the classical probability theory provides the appropriate framework for taking expectations, when probabilities are represented by intervals, one has to resort to the theory of capacities[3] in order to take expectations and to price a derivative (see e.g. Cherubini (1997)).

In illiquid markets, where the PCP fails to hold, uniqueness of the artificial probabilities leaves room to an interval. In our model, neither the artificial probability measure is unique, nor the process of the underlying asset takes precise values since we have aggregated partial and conflicting information coming from call and put prices. It follows that the theory of capacities cannot be used straightforward and we have to extend it in order to take into account interval values for the underlying asset.

In order to give an intuitive representation of the issue, we begin by taking a one period binomial model, i.e. two dates 0,1 and two states X={U, D}. Figure 4 illustrates our problem. The stock price at time zero is S and it takes interval values in both states at time one: $[\underline{Su},\overline{Su}]$ in state up, and $[\underline{Sd},\overline{Sd}]$ in state down, where $\underline{Su}$ and $\underline{Sd}$ are the lower bounds of the intervals, $\overline{Su}$ and $\overline{Sd}$ the upper bounds.

[3] Capacities were first introduced by Schmeidler (1989) in individual decision theory, and used by Dow and Werlang (1992), in a portfolio selection model. In both papers capacities are used as a representation of individual behaviour, by contrast in this model they arise at an aggregate level from the different information carried by call and put prices.

The risk free rate is equal to r>0. A derivative is traded on the market and its payoff is a function of the underlying asset. Since the latter takes an interval value, the payoff for the derivative security in each state is an interval too, i.e. $f[\underline{Su}, \overline{Su}] = [\underline{fu}, \overline{fu}]$, $f[\underline{Sd}, \overline{Sd}] = [\underline{fd}, \overline{fd}]$.

We have to derive the derivative price at time zero, f_0. We first examine the case in which f is an increasing function of *S*.

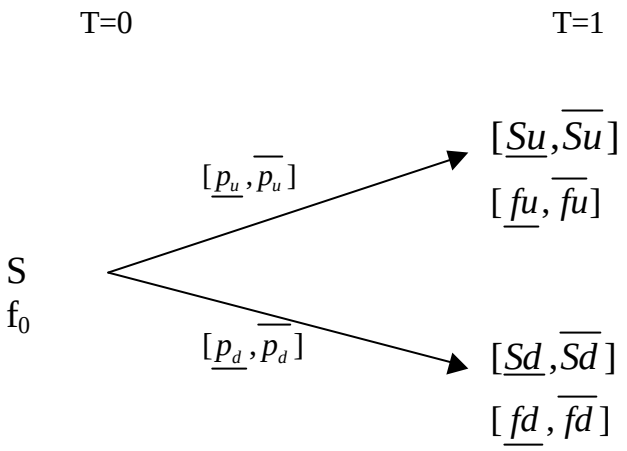

Figure 4. The binomial tree with interval values for the underlying asset.

Suppose that the artificial probability measure of an up move is included in a convex set $p_u \in [\underline{p_u}, \overline{p_u}]$ where $\underline{p_u} \le \overline{p_u}$. If we take a set function $m: P(X) \to [0,1]$ such that: $m(\varnothing) = 0, m(X) = 1$ and is monotone non decreasing with respect to set inclusion, and we set $m(U) = \underline{p_u}$ and $1 - m(D) = \overline{p_u}$, we can observe that μ is weakly subadditive, since $\underline{p_u} \le \overline{p_u}$ implies $m(U) + m(D) \le 1$. Such a set function is commonly called a capacity.

We can compute the boundaries of the expected value of the security on the set of probability measures in the following way:

$$\underline{f}_0 = \frac{1}{1+r} \min\left\{p_u[\underline{fu}, \overline{fu}] + (1-p_u)[\underline{fd}, \overline{fd}], p_u \in [\underline{p_u}, \overline{p_u}]\right\} = \frac{1}{1+r}[\underline{p_u}[\underline{fu}, \overline{fu}] + (1-\underline{p_u})[\underline{fd}, \overline{fd}]] \quad (25)$$

$$\overline{f}_0 = \frac{1}{1+r} \max\left\{p_u[\underline{fu}, \overline{fu}] + (1-p_u)[\underline{fd}, \overline{fd}], p_u \in [\underline{p_u}, \overline{p_u}]\right\} = \frac{1}{1+r}[\overline{p_u}[\underline{fu}, \overline{fu}] + (1-\overline{p_u})[\underline{fd}, \overline{fd}]] \quad (26)$$

Or equivalently, using the set function μ:

$$\underline{f_0} = \frac{1}{1+r}[[\underline{fd}, \overline{fd}] + m(U)([\underline{fu}, \overline{fu}] + [\underline{fd}, \overline{fd}])] = \frac{1}{1+r} E_*(f(S)) \quad (27)$$

$$\overline{f}_0 = \frac{1}{1+r}[[\underline{fu}, \overline{fu}] - m(D)([\underline{fu}, \overline{fu}] - [\underline{fd}, \overline{fd}])] = \frac{1}{1+r} E^*(f(S)) \quad (28)$$

where the operators E^* and E_* are known as upper and lower Choquet integral respectively.

Define $\overline{m}$ as the dual capacity of m such that $\overline{m}(U) = 1 - m(D)$, $\overline{m}(D) = 1 - m(U)$, $\overline{m}$ is superadditive, i.e. $\overline{m}(U) + \overline{m}(D) \geq 1$. Using $\overline{m}$ in equation (28) we obtain:

$$\overline{f}_0 = \frac{1}{1+r}[[\underline{fd}, \overline{fd}] + \overline{m}(U)([\underline{fu}, \overline{fu}] - [\underline{fd}, \overline{fd}])] = \frac{1}{1+r} E_*(f(S), \overline{m}) \tag{29}$$

The upper Choquet integral with respect to a subadditive capacity may be computed as the lower Choquet integral with respect to the dual superadditive capacity.

Suppose that the probability measure of a down move is included in a convex set $p_d \in [\underline{p_d}, \overline{p_d}]$ where $\underline{p_d} \leq \overline{p_d}$. If we set $1 - \overline{m}(U) = \underline{p_d}$ and $\overline{m}(D) = \overline{p_d}$, then we get:

$$\underline{f_0} = \frac{1}{1+r}[(1 - \overline{p_d})[\underline{fu}, \overline{fu}] + \overline{p_d}[\underline{fd}, \overline{fd}]] \tag{30}$$

$$\overline{f}_0 = \frac{1}{1+r}[(1 - \underline{p_d})[\underline{fu}, \overline{fu}] + \underline{p_d}[\underline{fd}, \overline{fd}]] \tag{31}$$

As the underlying asset takes interval values in state up and down, it follows that both the lower and the upper Choquet integral for the derivative security are themselves intervals, i.e. $\underline{f_0} \in [\underline{f_{0*}}, \underline{f_0}^*]$ and $\overline{f_0} \in [\overline{f_{0*}}, \overline{f_0}^*]$, where:

$$\underline{f_{0*}} = \frac{1}{1+r}[(1 - \overline{p_d})\underline{fu} + \overline{p_d}\,\underline{fd}] \tag{32}$$

$$\underline{f_0}^* = \frac{1}{1+r}[(1 - \overline{p_d})\overline{fu} + \overline{p_d\, fd})] \tag{33}$$

$$\overline{f}_{0*} = \frac{1}{1+r}[(1 - \underline{p_d})\underline{fu} + \underline{p_d}\,\underline{fd}] \tag{34}$$

$$\overline{f}_0^* = \frac{1}{1+r}[(1 - \underline{p_d})\overline{fu} + \underline{p_d}\,\overline{fd}] \tag{35}$$

Since, $\underline{f_{0*}} \leq \underline{f_0}^* \leq \overline{f}_{0*} \leq \overline{f_0}^*$ it follows that the largest bound of the derivative price is $[\underline{f_{0*}}, \overline{f_0}^*]$.

Analogously, if the derivative payoff is a decreasing function f of the underlying asset S, i.e. its payoff is higher in state down than in state up, its price bounds at time zero are computed using the lower and the upper bound respectively of the probability of state down (where the derivative has higher payoff):

$$\underline{f_{0*}} = \frac{1}{1+r}[(1 - \underline{p_d})\underline{fu} + \underline{p_d}\,\underline{fd}] \tag{36}$$

$$\overline{f_0}^* = \frac{1}{1+r}[(1 - \overline{p_d})\overline{fu} + \overline{p_d\, fd})] \tag{37}$$

It is easy to show that these are the largest bounds. This observation leads to the following proposition.

Proposition 1. The expected value of a derivative whose payoff is an increasing (decreasing) function of the underlying asset that takes interval values in state up $[\underline{Su},\overline{Su}]$ and in state down $[\underline{Sd},\overline{Sd}]$ when the artificial probability measure of an up move is included in a convex set $p_u \in [\underline{p_u},\overline{p_u}]$ where $\underline{p_u} \le \overline{p_u}$, is a bounded interval $[\underline{f_{0*}},\overline{f_0}^*]$, where $\underline{f_{0*}}$ is computed evaluating the lower (upper) Choquet integral of the lower bound of the derivative payoff in each state with respect to the subadditive capacity, discounted at the risk free rate and $\overline{f_0}^*$ is computed as the upper (lower) Choquet integral of the upper bound of the derivative payoff in each state with respect to the subadditive capacity, discounted at the risk free rate.

5. PRICING A DERIVATIVE

In this section we explain how to compute a derivative price on the PC-implied tree and how to use the results obtained.

In a n-period tree, the expected value of a derivative security can be computed by backward induction. Decomposing the entire n-period tree in (n(n+1))/2 binomial sub-trees, we compute by means of Proposition 1 the derivative value at each node (i,j), with i=1,...,j+1; j=0,...,n-1 until the initial node is reached, i.e. the current price of the derivative is found.

In order to give an example, we compute the price of a derivative whose payoff is an increasing function of the underlying asset payoff e.g. a call option, with strike price K. Figure 5 illustrates the case. Note that at time two, because of the centring condition, the underlying asset takes a precise value; we can always consider a crisp value as a collapsed interval.

Starting from the terminal nodes, we compute the call prices at time one as follows:

$$\underline{f}(2,1) = \frac{1}{1+r}[\underline{pu}(2,2)(\underline{S}(3,2) - K) + (1 - \underline{pu}(2,2))(S(2,2) - K)]$$

$$\overline{f}(2,1) = \frac{1}{1+r}[\overline{pu}(2,2)(\overline{S}(3,2) - K) + (1 - \overline{pu}(2,2))(S(2,2) - K)]$$

$$\underline{f}(1,1) = \frac{1}{1+r}[\underline{pu}(1,2)(S(2,2) - K) + (1 - \underline{pu}(1,2))(\underline{S}(1,2) - K)]$$

$$\overline{f}(1,1) = \frac{1}{1+r}[\overline{pu}(1,2)(S(2,2) - K) + (1 - \overline{pu}(1,2))(\overline{S}(1,2) - K)]$$

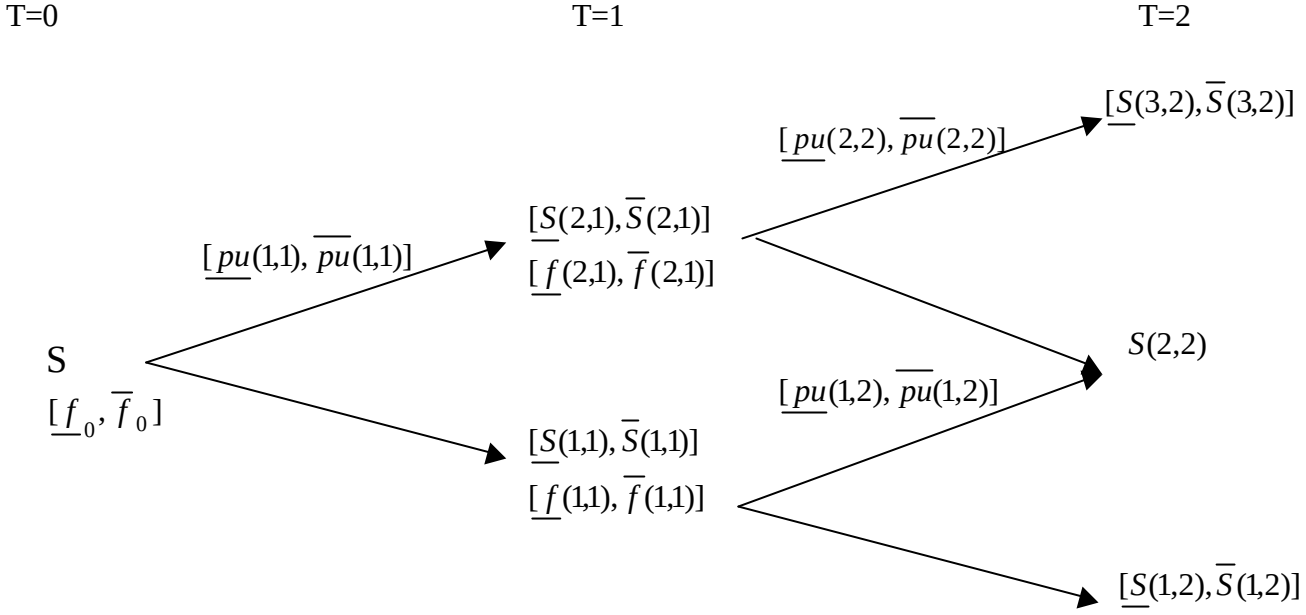

Figure 5. A two periods PC-implied tree.

And finally the call price at time zero is:

$$\underline{f}_0 = \frac{1}{1+r}[\underline{pu}(1,1)(\underline{f}(2,1)) + (1-\underline{pu}(1,1))(\underline{f}(1,1))]$$

$$\overline{f}_0 = \frac{1}{1+r}[\overline{pu}(1,1)(\overline{f}(2,1)) + (1-\overline{pu}(1,1))(\overline{f}(1,1))]$$

If we price a put option with strike price $K > \overline{S}(3,2)$, then we get:

$$\underline{f}(2,1) = \frac{1}{1+r}[\overline{pu}(2,2)(K-\overline{S}(3,2)) + (1-\overline{pu}(2,2))(K-S(2,2))]$$

$$\overline{f}(2,1) = \frac{1}{1+r}[\underline{pu}(2,2)(K-\underline{S}(3,2)) + (1-\underline{pu}(2,2))(K-S(2,2))]$$

$$\underline{f}(1,1) = \frac{1}{1+r}[\overline{pu}(1,2)(K-S(2,2)) + (1-\overline{pu}(1,2))(K-\overline{S}(1,2))]$$

$$\overline{f}(1,1) = \frac{1}{1+r}[\overline{pu}(1,2)(K-S(2,2)) + (1-\overline{pu}(1,2))(K-\underline{S}(1,2))]$$

And finally the put price at time zero is:

$$\underline{f}_0 = \frac{1}{1+r}[\overline{pu}(1,1)(\underline{f}(2,1)) + (1-\overline{pu}(1,1))(\underline{f}(1,1))]$$

$$\overline{f}_0 = \frac{1}{1+r}[\underline{pu}(1,1)(\overline{f}(2,1)) + (1-\underline{pu}(1,1))(\overline{f}(1,1))]$$

If we draw a comparison with the classical Derman and Kani model, a few comments are in order. It has been detected, see e.g. Chesney, Gibson and Loubergé (1995) and Cavallo and Mammola (2000), that the volatility implied by call prices is generally lower then the one implied by put

prices. If this is the case, the Derman and Kani tree, being constructed using call prices for the upper half and put prices for the lower half, give a higher volatility to lower nodes then it does for upper nodes. As a consequence, the classical Derman and Kani model overprices derivatives whose payoff is a decreasing function of the underlying asset and underprices derivatives whose payoff is an increasing function of the underlying asset.

On the other hand, if the volatility implied by call prices is higher than the one implied by put prices, the Derman and Kani implied tree overprices derivatives whose payoff is an increasing function of the underlying asset and underprices derivatives whose payoff is a decreasing function of the underlying asset.

In our model, the interval of prices for the derivative security better reflects the information available on the market regarding the underlying process.

The PC-implied tree can be used in markets where PCP deviations have been observed, to value illiquid options or other derivatives that are not traded in the market (for example, the derivative part of a structured note issued by an intermediary). The price interval obtained, that arises because of the market illiquidity, represents a bid-ask price quoted by an intermediary. To make this point more clear, let us analyse gains and losses of an intermediary that issues a derivative on the underlying asset whose process is represented by the PC-implied tree in Figure 4. In order to make a positive profit, the intermediary quotes the following prices: she buys the derivative at the lowest price and sells it at the highest.

Let us assume the derivative is an increasing function of the underlying asset. The gain from buying one unit of the derivative at a price P, is at least equal to the expected value of the derivative payoff minus the price P, i.e it is at least equal to:

$$\underline{f}_{0*} - P = \frac{1}{1+r}[\underline{p_u}\,\underline{fu} + (1-\underline{p_u})\,\underline{fd}] - P$$

and at most equal to:

$$\overline{f}_0^{\,*} - P = \frac{1}{1+r}[\overline{p}_u\,\overline{fu} + (1-\overline{p}_u)\,\overline{fd})] - P\,.$$

The gain from selling the derivative at a price P is equal to the price P minus the expected value of the derivative payoff that the writer has to pay to the holder, i.e. it is as least equal to:

$$P - \overline{f}_0^{\,*} = P - \frac{1}{1+r}[\overline{p}_u\,\overline{fu} + (1-\overline{p}_u)\,\overline{fd})]$$

and at most equal to:

$$P - \underline{f}_{0*} = P - \frac{1}{1+r}[\underline{p_u}\,\underline{fu} + (1-\underline{p_u})\,\underline{fd}]\,.$$

It follows that the intermediary is willing to buy the derivative if price P is less then $\underline{f}_{0*}$ and is willing to sell it if its price P is more then $\overline{f}_0^*$. At prices between the bounds, then there is no incentive for the intermediary to issue the derivative[4]. This situation is illustrated in Figure 6. Thus the derivative price bounds represent the bid-ask prices quoted by the intermediary that has issued the security.

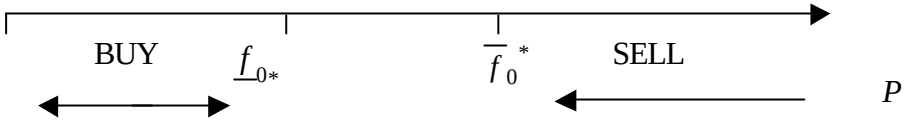

Figure 6. The bid-ask price for the derivative security.

6. CONCLUSIONS

In this paper we have proposed and implemented a procedure to construct implied trees in illiquid markets and we have extended the Choquet integral definition in order to take into account interval values for the stock payoff. Our model accounts for violations of the Put Call parity in illiquid markets and exploits all the information contained in such deviations.

When the Put Call parity fails to hold, uniqueness of the artificial probabilities leaves room to an interval. In order to bound the artificial probabilities and the underlying stock prices at each node of the tree, we have derived two implied trees: one using only call options and the other one using only put options. The implied tree we obtain incorporates all the information in call and put prices.

In order to use the PC-tree to take expectations and thus value any derivative on the underlying stock, we have to resort to the use of capacities. To this end we have extended the Choquet integral in order to take into account interval values for the stock payoff, that arise because of the conflicting information coming from call and put prices.

The PC-implied tree can be used in markets where PCP deviations have been observed, to value illiquid options or other derivatives that are not traded in the market (for example, the derivative part of a structured note issued by an intermediary). The price interval obtained represents a bid-ask price quoted by an intermediary.

[4] Dow and Werlang (1992) explained the portfolio inertia puzzle using a non additive probability measure to represent agents' preferences. Differently from their approach, in our model the capacity is derived from the conflicting information of call and put prices, at a market level and the stock payoffs take imprecise values.

Future research work includes, from an empirical point of view, a validation of the model proposed and from a theoretical one, an alternative derivation of the PC-implied tree.

Specifically, we believe that the contemporary use of both a call and a put option at each node of the tree to imply the next level variables would allow a more accurate estimate of the future variables. However, it should be stressed that this extension would lead to a substantial increase of the computational complexity and therefore it is suitable only for trees with a few levels.

REFERENCES

Black F., Scholes M. (1973) "The pricing of options and corporate liabilities" *Journal of Political Economy,* 81, 637-654.

Cavallo L., Mammola P. (2000) "Empirical tests of efficiency of the Italian index options market" *Journal of Empirical Finance,* 7, 173-193.

Chesney M., Gibson R., Loubergé H. (1995) "Arbitrage trading and index option pricing at SOFFEX: an empirical study using daily and intradaily data" *Finanzmarkt und Portfolio Management,* 1, 35-60.

Cherubini U.(1997) "Fuzzy measures and asset prices" *Applied Mathematical Finance,* 4, 135-149

Choquet G. (1955) "Theory of Capacities" *Annales de l'Institut Fourier,* 5, 131-295.

Cox J., Ross S. and Rubinstein M. (1979) "Option pricing, a simplified approach" *Journal of Financial Economics,* 7, 229-263.

Dempster A.P. (1967) "Upper and Lower probabilities induced by a multivalued mapping" *Annals of Mathematical Statistics,* 38, 325-339.

Derman E., Kani I. (1994) "Riding on a smile" *Risk,* 7, (2), 32-39.

Dow J., Ribeiro Da Costa Werlang S.(1992) "Uncertainty aversion, risk aversion and the optimal choice of portfolio" *Econometrica,* 60, 197-204.

Evnine J., Rudd A. (1985) "Index options, the early evidence" *Journal of Finance,* 40, 743-756.

Gilboa I. (1987) "Expected utility theory with purely subjective non additive probabilities" *Journal of Mathematical Economics,* 16, 65-88.

Gould J.P., Galai D. (1974) "Transaction costs and the relationship between put and call prices" *Journal of Financial Economics,* 1,105-129.

Hull J., White A. (1987) "The pricing of options on assets with stochastic volatilities" *Journal of Finance,* 42, 281-300

Jarrow, R.A. (1988) *Finance Theory,* Prentice-Hall International Editions, New York.

Kamara A., Miller T.W. Jr. (1995) "Daily and intradaily tests of European put call parity", *Journal*

of Financial and Quantitative Analysis, 30, (4), 519-539.

Klemkosky R.C., Resnick B.G. (1979) "Put call parity and market efficiency" *The Journal of Finance*, 34, (5), 1141-1155.

Merton R. (1976) "Options pricing when underlying stock returns are discontinuous" *Journal of Financial Economics*, 3, 125-144.

Mittnik S., Rieken S. (2000) "Put –call parity and the informational efficiency of the German DAX index options market" *International Review of Financial Analysis*, 9, 259-279.

Moore R.E. (1979*) Methods and applications of interval analysis* SIAM Studies in Applied Mathematics.

Nelson D., B., Ramaswamy K. (1990) "Simple binomial processes as diffusion approximations in financial models" *The Review of Financial Studies*, 3, (3), 393-430.

Nisbet M. (1992) "Put-call parity theory and an empirical test of the efficiency of the London Traded Options Market " *Journal of Banking and Finance*, 16, 381-403.

Rendleman R.J. Jr., Bartter B.J. (1979) "Two states option pricing" *Journal of Finance*, 34, (5), 1093-1111.

Rubinstein M. (1994) "Implied binomial trees" *Journal of Finance*, 64, (3), 770-818.

Shimko D. (1993) "Bounds of probability" *Risk*, 6, 33-37.

Scmeidler D. (1989) "Subjective probability and expected utility without additivity" *Econometrica*, 57, (3), 571-587.

Stapleton R.C., Subrahmanyam M.G. (1984) "The valuation of options when asset returns are generated by a binomial process" *Journal of Finance*,39, (5) 1525-1539.

Stoll H.R. (1969) "The relationship between put and call options prices" *Journal of Finance* 24, 801-824.

Strong N., Xu X. (1999) "Do S&P500 index options violate the martingale restriction?" *Journal of Futures markets*, 499-521.