A note on fuzzy linear systems

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Abstract

The aim of this paper is to analyse the solution of a fuzzy system when the classical solution based on standard fuzzy mathematics fails to exist. In particular we analyse the solution of the system Ax=b with A squared matrix with positive fuzzy coefficients and y crisp vector of positive elements. This system is particularly important for financial applications. We propose two different solution methods that are based respectively on the work of Buckley et al. (2002) and Friedman, Ming and Kandel (1998). An application to an important financial problem, the derivation of the artificial probabilities in a lattice framework, is provided.

Keywords: fuzzy linear systems, fuzzy numbers, binomial tree.

1. Introduction

Various financial problems boil down to the solution of linear systems of equations. When the estimation of the system parameters is difficult, it is convenient to represent some of the system parameters with fuzzy numbers rather than crisp numbers.

Let $Ax=y$ be a fuzzy linear system, where A is a square matrix of fuzzy coefficients a_{ij} , $i= 1, ..., n, j = 1, ..., n$ and y is a fuzzy vector.

$$A = \begin{bmatrix} a_{1,1} & .... & a_{1,n} \\ : & ... & : \\ a_{n,1} & ... & a_{n,n} \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ : \\ x_n \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ : \\ y_n \end{bmatrix} \qquad (1)$$

The alpha-cut of a fuzzy number $a_{i,j}$ is denoted by: $a_{i,j}(\alpha) = [\underline{a}_{i,j}, \overline{a}_{i,j}]$, where, for brevity, the dependence of the bounds of the interval on α is omitted. If $a_{i,j}$ is a real number, then the alpha-cut $a_{i,j}(\alpha) = a_{i,j}, \forall \alpha \in [0,1]$.

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In this paper we investigate the solution of a fuzzy system when the classical solution based on standard fuzzy mathematics fails to exist.

Using the α-cut representation, we write the i[th] equation of the system as follows:

$$\sum_{j=1}^{n}[\underline{a}_{i,j},\overline{a}_{i,j}][\underline{x}_j,\overline{x}_j]=[\underline{y}_i,\overline{y}_i] \tag{2}$$

Using interval addition and multiplication (see Appendix 1) to evaluate the left hand side of equation (2) we get:

$$\sum_{j=1}^{n}[\underline{a_{i,j}x_j},\overline{a_{i,j}x_j}]=[\underline{y}_i,\overline{y}_i]$$

and in turn:

$$\begin{cases}\sum_{j=1}^{n}\underline{a_{i,j}x_j}=\underline{y}_i\\ \sum_{j=1}^{n}\overline{a_{i,j}x_j}=\overline{y}_i\end{cases}$$

where $\underline{a_{i,j}x_j}=\min(\underline{a}_{i,j}\underline{x}_j,\overline{a}_{i,j}\overline{x}_j,\overline{a}_{i,j}\underline{x}_j,\underline{a}_{i,j}\overline{x}_j)$, $\overline{a_{i,j}x_j}=\max(\underline{a}_{i,j}\underline{x}_j,\overline{a}_{i,j}\overline{x}_j,\overline{a}_{i,j}\underline{x}_j,\underline{a}_{i,j}\overline{x}_j)$.

This implies that to solve one fuzzy equation (a system of *n* fuzzy equations) we have to solve a system of two (a system of *2n*) non fuzzy equations.

By making hypothesis on the sign of $\underline{x}_j,\overline{x}_j$, we solve for the $\underline{x}_j,\overline{x}_j$ and hope they produce the alpha-cuts of a fuzzy number x_j. As shown in [2] too often the system has no solution.

A particular fuzzy linear system that has no solution using standard operations between fuzzy numbers is the system $Ax=y$ where A is a square matrix of fuzzy coefficients a_{ij} , $i= 1, \ldots, n$, $j = 1, \ldots, n$ and y is a vector of crisp terms. Recalling that y is a crisp vector:

$$\begin{cases}\sum_{j=1}^{n}\underline{a_{i,j}x_j}=y_i\\ \sum_{j=1}^{n}\overline{a_{i,j}x_j}=y_i\end{cases} \tag{3}$$

as $a_{i,j}$ is a fuzzy number, system (3) has trivially no solution using regular operations between fuzzy numbers.

Two are the alternative solutions proposed in the literature to overcome the difficulty of finding a solution using regular operations between fuzzy numbers: the Buckley et al. (2002) solution and the Friedman, Ming and Kandel (1998) solution.

Buckley et al. (2002) handle the general problem of constructing a solution for the fuzzy matrix equation $Ax=y$ when the elements in A and y are triangular fuzzy numbers (TFN). They propose a solution strategy that involves the use of three different methods that should be investigated sequentially. They start from the most precise method and, if it is too difficult to compute, they use the second best method. If the second method is also hard to implement, they resort to the third method, that is the easiest.

Friedman, Ming and Kandel (1998) analyse a system *Ax=y* where *A* is the matrix with crisp coefficients and *y* is the vector of fuzzy coefficients. They introduce the concept of a weak fuzzy solution, that is used whenever the classical solution does not exist.

In Friedman, Ming and Kandel (2000), the authors investigate the important issue of duality. As they recall, in general there is no inverse element for a fuzzy number *u*, i.e. an element such that *u+v=0*, therefore the fuzzy linear system *(A-B) x = y* can not be replaced by the fuzzy linear system *Ax = Bx + y* (the dual fuzzy linear system). In line with their observation we point out that any shift of one term from one side to the other of the equality may lead to a different fuzzy system, as it is illustrated in the following.

Consider the following *2x2* fuzzy system: *Ax = y*,

$$\begin{cases} a_{11}(\alpha)x_1 + a_{12}(\alpha)x_2 = y_1(\alpha) \\ a_{21}(\alpha)x_1 + a_{22}(\alpha)x_2 = y_2(\alpha) \end{cases} \tag{4}$$

by writing the system in terms of α-cuts we get:

$$\begin{cases} [\underline{a}_{11}, \overline{a}_{11}]x_1 + [\underline{a}_{12}, \overline{a}_{12}]x_2 = [\underline{y}_1, \overline{y}_1] \\ [\underline{a}_{21}, \overline{a}_{21}]x_1 + [\underline{a}_{22}, \overline{a}_{22}]x_2 = [\underline{y}_2, \overline{y}_2] \end{cases} \tag{5}$$

Suppose that A^{-1} exists for all $a_{ij} \in a(\alpha), \forall \alpha \in [0,1]$ and, for simplicity, that the unknowns in system 5 are all positive. Any shift of one term from one side of the equality to the other will lead to a different crisp system, e.g. system 5 leads to:

$$\begin{cases} \underline{a}_{11}\underline{x}_1 + \underline{a}_{12}\underline{x}_2 = \underline{y}_1 \\ \underline{a}_{21}\underline{x}_1 + \underline{a}_{22}\underline{x}_2 = \underline{y}_2 \\ \overline{a}_{11}\overline{x}_1 + \overline{a}_{12}\overline{x}_2 = \overline{y}_1 \\ \overline{a}_{21}\overline{x}_1 + \overline{a}_{22}\overline{x}_2 = \overline{y}_2 \end{cases}$$

If we change system 5, by shifting one term to the right hand side, e.g.:

$$\begin{cases} [\underline{a}_{11}, \overline{a}_{11}]x_1 = [\underline{y}_1, \overline{y}_1] - [\underline{a}_{12}, \overline{a}_{12}]x_2 \\ [\underline{a}_{21}, \overline{a}_{21}]x_1 + [\underline{a}_{22}, \overline{a}_{22}]x_2 = [\underline{y}_2, \overline{y}_2] \end{cases} \tag{6}$$

the crisp system changes as follows:

$$
\begin{cases}
\underline{a}_{11}\underline{x}_1 + \overline{a}_{12}\overline{x}_2 = \underline{y}_1 \\
\underline{a}_{21}\underline{x}_1 + \underline{a}_{22}\underline{x}_2 = \underline{y}_2 \\
\overline{a}_{11}\overline{x}_1 + \underline{a}_{12}\underline{x}_2 = \overline{y}_1 \\
\overline{a}_{21}\overline{x}_1 + \overline{a}_{22}\overline{x}_2 = \overline{y}_2
\end{cases}
$$

Depending on how we write the system, for each parameter we may use the lower or the upper bound of the interval. Moreover, depending on the hypothesis that we make on the sign of each unknown, we should use different combinations of the bounds of the parameters and of the unknowns. As a consequence, we may find different solutions for the same fuzzy system, depending on how we write each equation of the system.

The aim of this paper is twofold. First we propose two different methods, based respectively on the solution concepts of Buckley et al. (2002) and Friedman, Ming and Kandel (1998), that find a solution that does not change depending on how we write each equation of the system. Second, we provide an application of the methods proposed to an important financial problem: the derivation of the artificial probabilities in a lattice framework.

The plan of the paper is the following: in section 2 we briefly illustrate the Buckley et al. (2002) solution for the fuzzy matrix equation $Ax=y$ when the elements in A and y are TFN, highlighting the main advantages and disadvantages. In section 3 we propose the first method to find the largest solution interval, that is based on the solution concept of Buckley et al. (2002). In section 4 we illustrate the Friedman, Ming and Kandel (1998) solution and we explain the relation among their solution, the solution proposed in section 3 and the classical solution. In section 5 we propose the second method to find the largest solution interval that involves an extension of the Friedman, Ming and Kandel (1998) method to a matrix with fuzzy coefficients. In section 6 we present an application of the two different methods to the financial problem: the derivation of the artificial probabilities in a lattice framework. The last section concludes.

2. Buckley et al. (2002) solution

Buckley et al. (2002) handle the general problem of constructing a solution interval for the fuzzy matrix equation $Ax=y$ when the elements of the matrix A, $a_{i,j,}$ and the elements of the vector y, y_i , $i=1,\ldots,n$ $j=1,\ldots,n$ are TFN.

Define $a(\alpha)=\prod_{i,j=1}^{n} a_{i,j}(\alpha)$, $y(\alpha)=\prod_{i,j=1}^{n} y_{i,j}(\alpha)$, $\forall \alpha \in [0,1]$. Let $v=(a_{1,1},...,a_{n,n}) \in R^{n^2}$ be a vector in $a(0)$, that determines a crisp matrix and let $y=(y_1,...,y_n) \in R^n$ be a crisp vector in *y(0)*. Assume that A^{-1} exists $\forall v \in a(0)$.

They propose three different solutions: the first X_J, investigates the joint solution and then the marginals for each unknown, the second and the third solutions, X_E and X_I investigate directly the marginals, and are based respectively on the extension principle and fuzzy arithmetic.

The joint solution X_J is a fuzzy subset of R^n defined as follows:

$$X_J(\alpha)=\{x \mid Ax=y, v \in a_{i,j}(\alpha), y_i \in y_i(\alpha)\}.$$

The marginals X_{Ji} are obtained by projecting X_J onto the coordinate axes:

$X_{Jj}(w)=\max\{X_J(x) \mid x \in R^n, x_j=w\}$ for $j = 1, ..., n$.

The second and the third solutions, X_E and X_I investigate directly the marginals, that are founded by using Cramer's rule to solve for each unknown:

$$x_j=\frac{|A_j|}{|A|}, \qquad j=1,...,n \tag{7}$$

where A is the crisp matrix determined by v, A_j has its j-th column replaced by y and fuzzify it by using or the extension principle, or interval arithmetic.

The second solution, X_E, is investigated by using the extension principle:

$$X_{Ej}=\max\{\pi(v,y) \mid x_j=|A_j|/|A|\}$$

If $X_{Ej}(\alpha)=[\underline{x}_{Ej},\overline{x}_{Ej}]$ then

$$\underline{x}_{Ej}=\min\left\{\frac{|A_j|}{|A|} \mid v \in a(\alpha), y \in y(\alpha)\right\}$$

$$\overline{x}_{Ej}=\max\left\{\frac{|A_j|}{|A|} \mid v \in a(\alpha), y \in y(\alpha)\right\}$$

The third solution, $X_{Ij}(\alpha)=[\underline{x}_{Ij},\overline{x}_{Ij}]$, is obtained by fuzzifying ex-post the crisp solution in equation 7, using interval arithmetic.

They show that: $X_C \le X_J \le X_E \le X_I$ where X_C is the classical solution. They propose to use X_C if it exists, if the classical solution do not exist use X_J, if the joint solution is too difficult to investigate use X_E , if also X_E is difficult to evaluate, then use X_I.

In all the cases in which it is difficult to investigate at least X_E, they suggest to simply fuzzify ex-post the crisp solution to the system. The main drawback of such a choice is that the solution bounds do not have any crisp system that supports them (as it is illustrated in the example in section 6). Instead, it is desirable that any crisp value that belongs to the solution interval is obtained by using a crisp value $a_{ij} \in a_{i,j}(\alpha)$ and $y_i \in y_i(\alpha)$ at the same level of uncertainty.

3. An alternative method to find the solution

In the following we propose a practical algorithm that finds directly the marginals for each unknown and overcomes the drawback of X_I. It has the advantage to be easily implementable and to take into account the important issue of duality.

Consider the following *2x2* fuzzy system: $Ax = y$,

$$\begin{cases} a_{11}(\alpha)x_1 + a_{12}(\alpha)x_2 = y_1(\alpha) \\ a_{21}(\alpha)x_1 + a_{22}(\alpha)x_2 = y_2(\alpha) \end{cases}$$

by writing the system in terms of α-cuts we get:

$$\begin{cases} [\underline{a}_{11},\overline{a}_{11}]x_1 + [\underline{a}_{12},\overline{a}_{12}]x_2 = [\underline{y}_1,\overline{y}_1] \\ [\underline{a}_{21},\overline{a}_{21}]x_1 + [\underline{a}_{22},\overline{a}_{22}]x_2 = [\underline{y}_2,\overline{y}_2] \end{cases}$$

Recall that it is supposed that A^{-1} exists for all $a_{ij} \in a(\alpha), \forall \alpha \in [0,1]$.

We are looking for the solution vector x that, for each α satisfy $Ax=y$ where $a_{i,j} \in a_{i,j}(\alpha), y_i \in y_i(\alpha)$. It follows that any crisp value $a_{i,j} \in a_{i,j}(\alpha), y_i \in y_i(\alpha)$,determines a crisp solution that should belong with membership α to the fuzzy solution x.

Using Cramer's rule to find the solutions of a crisp system $Ax = y$, we note that x_j is an increasing or decreasing function of each $a_{i,j} \in A$ and of each $y_i \in y$. It follows that in the fuzzy system where $a_{i,j} \in A$ and y_i are fuzzy numbers, and $a_{i,j}(\alpha)$ and $y_i(\alpha)$ are the alpha-cuts of the fuzzy number, the bounds of the solution interval for each unknown $x_j(\alpha)$ should be investigated by using each bound of the $a_{i,j}(\alpha)$ and $y_i(\alpha)$.

Therefore we have to solve 2^6 systems (for each parameter we can choose to use the lower bound or the upper bound of the interval), e.g. one of the 2^6 systems is:

$$\begin{cases} \underline{a}_{11}x_1 + \overline{a}_{12}x_2 = \underline{y}_1 \\ \underline{a}_{21}x_1 + \underline{a}_{22}x_2 = \overline{y}_2 \end{cases}$$

The final solution is investigated by taking the minimum and the maximum of the solutions found in each system for each unknown:

$\underline{x}_1 = \min(x_1(system1),...,x_1(system2^6))$

$\overline{x}_1 = \max(x_1(system1),...,x_1(system2^6))$

$\underline{x}_2 = \min(x_2(system1),...,x_2(system2^6))$

$\overline{x}_2 = \max(x_2(system1),...,x_2(system2^6))$.

This procedure ensures that we are taking all the possible solutions consistent with the parameters of the system, but it does not guarantee that the solutions for x_1 and x_2 are fuzzy numbers. Thus, an ex-post check is needed in order to exclude the solutions that are not fuzzy numbers. If $x_1(1) \notin [\underline{x}_1(0), \overline{x}_1(0)]$ or $x_2(1) \notin [\underline{x}_2(0), \overline{x}_2(0)]$, then we conclude that there is no solution to the system.

A simplification of the previous method, is to find the solutions for $\alpha = 1$ and $\alpha = 0$ and impose ex post a triangular form on the solution, whenever $x_1(1) \in [\underline{x}_1(0), \overline{x}_1(0)]$ and $x_2(1) \in [\underline{x}_2(0), \overline{x}_2(0)]$.

In order to find $x_1(1)$, $x_2(1)$, we just solve the crisp system, substituting α=1 in the fuzzy system.

In order to find $x_1(0) = [\underline{x}_1(0), \overline{x}_1(0)]$, $x_2(0) = [\underline{x}_2(0), \overline{x}_2(0)]$, we apply the previous methodology, using instead of $\underline{a}_{i,j}$, $\overline{a}_{i,j}, \underline{y}_i, \overline{y}_i$ that depend on α, their crisp values $\underline{a}_{i,j}(0)$, $\overline{a}_{i,j}(0), \underline{y}_i(0), \overline{y}_i(0)$.

If $x_1(1) \in [\underline{x}_1(0), \overline{x}_1(0)]$ and $x_2(1) \in [\underline{x}_2(0), \overline{x}_2(0)]$, then we take as solution the two triangular fuzzy numbers $x_1 = (\underline{x}_1(0), x(1), \overline{x}_1(0))$ and $x_2 = (\underline{x}_2(0), x(1), \overline{x}_2(0))$, otherwise we conclude that there is no solution to the system.

4. Friedman, Ming and Kandel (1998) solution

Friedman, Ming and Kandel (1998) analyse a system Ax=y where A is the matrix with crisp coefficients and y is the vector of fuzzy coefficients and in order to find a solution, they use the conventional rules of addition and multiplication of a real number and a fuzzy number, reported in Appendix 1. They rewrite the system $Ax = y$ as $Sx = y$:

$$S = \begin{bmatrix} s_{1,1} & \cdots & s_{1,n} & s_{1,n+1} & \cdots & s_{1,2n} \\ \vdots & . & & & . & \vdots \\ s_{n,1} & \cdots & . & . & \cdots & s_{n,2n} \\ s_{n+1,1} & \cdots & . & . & \cdots & s_{n+1,2n} \\ \vdots & . & & & . & \vdots \\ s_{2n,1} & \cdots & s_{2n,n} & s_{2n,n+1} & \cdots & s_{2n,2n} \end{bmatrix}, \qquad x = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ (-\overline{x}_1) \\ \vdots \\ (-\overline{x}_n) \end{bmatrix}, \qquad y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ (-\overline{y}_1) \\ \vdots \\ (-\overline{y}_n) \end{bmatrix}$$

where $s_{i,j}$ are determined as follows:

if $a_{i,j} \geq 0$ $s_{i,j} = a_{i,j}$, $s_{i+n,j+n} = a_{i,j}$,

if $a_{i,j} < 0$, $s_{i+n,j} = -a_{i,j}$, $s_{i,j+n} = -a_{i,j}$,

where $i = 1,...,n$ $j = 1,...,n$. Any s_{ij} not determined by the abovementioned conditions is equal to 0.

The structure of S is $S = \begin{bmatrix} B & C \\ C & B \end{bmatrix}$, they demonstrate that S is non-singular if and only if the matrices B-C and B+C are both non singular, and that if S^{-1} exists it has the form:

$$S^{-1} = \left[t_{i,j}\right] = \begin{bmatrix} D & E \\ E & D \end{bmatrix}.$$

Assuming that S^{-1} exists, the solution vector is $x = S^{-1}y$, $x = \left\{ \underline{x}_i, -\overline{x}_i, i = 1,...,n \right\}$, $\underline{x}_i \leq \overline{x}_i$, such that:

$$\begin{cases} \sum_{j=1}^{n} \underline{a_{i,j} x_j} = \underline{y}_i \\ \sum_{j=1}^{n} \overline{a_{i,j} x_j} = \overline{y}_i \end{cases} \tag{8}$$

If ex-post $\underline{x} \geq \overline{x}$, they define the fuzzy solution $u = \left\{ \underline{u}_i, \overline{u}_i, i = 1,...,n \right\}$ as:

$\underline{u}_i = \min \left\{ \underline{x}_i, \overline{x}_i, \underline{x}(1) \right\}$

$\overline{u}_i = \max \left\{ \underline{x}_i, \overline{x}_i, \underline{x}(1) \right\}$

By the use of $\underline{x}(1)$ they eliminate the possibility that a fuzzy number possess an angle greater than 90°, i.e. the possibility that the peak value is not contained in the support of the fuzzy number. In this way they artificially force a quantity that is not a fuzzy number to become a fuzzy number.

The solution x_i is a strong fuzzy solution if $\underline{x}_i = \min\left\{\underline{x}_i, \overline{x}_i, \underline{x}(1)\right\}$ and $\overline{x}_i = \max\left\{\underline{x}_i, \overline{x}_i, \underline{x}(1)\right\}$, otherwise u is a weak fuzzy solution, i.e. a solution where $\underline{x}_i \geq \overline{x}_i$.

The condition in order to have a strong fuzzy solution is: $\overline{x} - \underline{x} = \sum_{j=1}^{n} (t_{i,j} + t_{i,n+j})(\overline{y}_j - \underline{y}_j) \geq 0$.

Note that in the case of a weak fuzzy solution, system (8) is trivially not verified using standard rules of addition and multiplication.

As it is shown in the following, the procedure to solve the system of Friedman, Ming and Kandel (1998) is a special case of the classical method that uses standard rules of fuzzy addition and multiplication, the only difference is in the solution vector: in the case of a strong fuzzy solution the Friedman, Ming and Kandel (1998) solution is equivalent to the one of the standard approach, while in the case of a weak fuzzy solution, it is different since using the standard methodology we would have found no solution to the system.

The system $\begin{bmatrix} B & C \\ C & B \end{bmatrix}\begin{bmatrix} \underline{x} \\ -\overline{x} \end{bmatrix} = \begin{bmatrix} \underline{y} \\ -\overline{y} \end{bmatrix}$ is equivalent to the system $\begin{bmatrix} B & -C \\ -C & B \end{bmatrix}\begin{bmatrix} \underline{x} \\ \overline{x} \end{bmatrix} = \begin{bmatrix} \underline{y} \\ \overline{y} \end{bmatrix}$ which is the one that we obtain by applying the standard rules of fuzzy addition and multiplication.

Proof.

$$\begin{cases} \sum_{j=1}^{n} \min(a_{i,j}[\underline{x}_j, \overline{x}_j]) = \underline{y}_i \\ \sum_{j=1}^{n} \max(a_{i,j}[\underline{x}_j, \overline{x}_j]) = \overline{y}_i \end{cases}$$

by the rules of multiplication of a crisp number and a fuzzy number (see Appendix 1):

if $a_{i,j} \geq 0$ then:

$$\min(a_{i,j}[\underline{x}_j, \overline{x}_j]) = a_{i,j}\underline{x}_j$$

$$\max(a_{i,j}[\underline{x}_j, \overline{x}_j]) = a_{i,j}\overline{x}_j$$

if $a_{i,j} < 0$

$$\min(a_{i,j}[\underline{x}_j, \overline{x}_j]) = a_{i,j}\overline{x}_j$$

$$\max(a_{i,j}[\underline{x}_j, \overline{x}_j]) = a_{i,j}\underline{x}_j$$

Using these rules we rewrite the system, obtaining the system *S'x'=y'*

$$S' = \begin{bmatrix} B' & C' \\ C' & B' \end{bmatrix} \quad x' = \begin{bmatrix} \underline{x} \\ \overline{x} \end{bmatrix} \qquad y' = \begin{bmatrix} \underline{y} \\ \overline{y} \end{bmatrix}$$

where *B'* = *B* contains the positive $a_{i,j}$, and *C'* = -*C* contains the negative $a_{i,j}$.

$S'=[s_{i,j}]$ is in fact determined as follows:

if $a_{i,j} \geq 0$ $s_{i,j} = a_{i,j}$, $s_{i+n,j+n} = a_{i,j}$,

if $a_{i,j} < 0$, $s_{i+n,j} = a_{ij}$, $s_{i,j+n} = a_{i,j}$,

where $i = 1,...,n$ $j = 1,...,n$. Any s_{ij} not determined by the abovementioned conditions is equal to 0.

The system *S'x'=y'* is equivalent to the system: *Sx* = *y*, in fact:

$$\begin{cases} B\underline{x} - C\overline{x} = \underline{y} \\ -C\underline{x} + B\overline{x} = \overline{y} \end{cases} = \begin{cases} B\underline{x} + C(-\overline{x}) = \underline{y} \\ C\underline{x} + B(-\overline{x}) = (-\overline{y}) \end{cases}$$

Therefore in case of a strong fuzzy solution, the Friedman, Ming and Kandel (1998) solution coincides with the classical solution.

The Friedman, Ming and Kandel (1998) method yield to a solution interval that is in general not contained in the solution interval proposed in section 3 (see e.g. Appendix 2).

However if $a_{ij} \geq 0$ the solution interval of Friedman, Ming and Kandel (1998) is contained in the solution interval proposed in section 3.

Proof.

Let us consider for simplicity a crisp matrix *[a*$_{ij}$*] i =j = 2* According to the method proposed in section 3 in order to solve the system, each equation can be written in two different ways:

Equation 1) $a_{11}x_1 + a_{12}x_2 = \underline{y}_1$ or $a_{11}x_1 + a_{12}x_2 = \overline{y}_1$

Equation 2) $a_{21}x_1 + a_{22}x_2 = \underline{y}_2$ or $a_{21}x_1 + a_{22}x_2 = \overline{y}_2$

We thus have 4 different systems to be solved:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = \underline{y}_1 \\ a_{21}x_1 + a_{22}x_2 = \underline{y}_2 \end{cases} \tag{9}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = \underline{y}_1 \\ a_{21}x_1 + a_{22}x_2 = \overline{y}_2 \end{cases} \tag{10}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = \overline{y}_1 \\ a_{21}x_1 + a_{22}x_2 = \underline{y}_2 \end{cases} \tag{11}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = \overline{y}_1 \\ a_{21}x_1 + a_{22}x_2 = \overline{y}_2 \end{cases} \tag{12}$$

and the solution is:

$\underline{x}_1 = \min(x_1(system9), x_1(system10), x_1(system11), x_1(system12))$

$\overline{x}_1 = \max(x_1(system9), x_1(system10), x_1(system11), x_1(system12))$

$\underline{x}_2 = \min(x_2(system9), x_2(system10), x_2(system11), x_2(system12))$

$\overline{x}_2 = \max(x_2(system9), x_2(system10), x_2(system11), x_2(system12))$

The solution of Friedman, Ming and Kandel (1998) coincides with the solution interval given by systems (9) and (12) and it is thus contained in the solution interval proposed in section 3.

5. An extension of the Friedman, Ming and Kandel (1998) method to a matrix with fuzzy coefficients

In this section we extend the Friedman, Ming and Kandel (1998) methodology to a matrix with fuzzy coefficients. We restrict our attention to $A = [a_{i,j}]$ matrix with fuzzy coefficients $a_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}]$, where $\underline{a}_{i,j} \geq 0$ and y fuzzy vector of coefficients, with $y_i \geq 0$.

In line with Friedman Ming and Kandel, we define the fuzzy number vector $x = [\underline{x}, \overline{x}]$ solution of the system if:

$$\sum_{j=1}^{n} \underline{a_{i,j} x_j} = \underline{y}_i \text{ and } \sum_{j=1}^{n} \overline{a_{i,j} x_j} = \overline{y}_i .$$

By applying the rules of fuzzy numbers, we get:

$$\begin{cases} \sum_{j=1}^{n} \min([\underline{a}_{i,j}, \overline{a}_{i,j}][\underline{x}_j, \overline{x}_j]) = \underline{y}_i \\ \sum_{j=1}^{n} \max([\underline{a}_{i,j}, \overline{a}_{i,j}][\underline{x}_j, \overline{x}_j]) = \overline{y}_i \end{cases}$$

If $\underline{a}_{i,j} \geq 0$, depending on the sign of $\underline{x}_j$ and $\overline{x}_j$, $\min([\underline{a}_{i,j}, \overline{a}_{i,j}][\underline{x}_j, \overline{x}_j])$ is obtained by looking at one of cases 1), 2) or 3) of Appendix 1.

In the following we present only the case in which we make the hypothesis of a positive x_j, and the system is written as $Ax=y$, keeping in mind that the same procedure should be repeated for all the other cases.

By using standard rules of addition and multiplication among fuzzy numbers, we rewrite the system as follows:

$$S = \begin{bmatrix} s_{1,1} & \cdots & s_{1,n} & s_{1,n+1} & \cdots & s_{1,2n} \\ \vdots & \ddots & & & \ddots & \vdots \\ s_{n,1} & \cdots & \cdot & \cdot & \cdots & s_{n,2n} \\ s_{n+1,1} & \cdots & \cdot & \cdot & \cdots & s_{n+1,2n} \\ \vdots & \ddots & & & \ddots & \vdots \\ s_{2n,1} & \cdots & s_{2n,n} & s_{2n,n+1} & \cdots & s_{2n,2n} \end{bmatrix}, \qquad x = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ \overline{x}_1 \\ \vdots \\ \overline{x}_n \end{bmatrix}, \qquad y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ \overline{y}_1 \\ \vdots \\ \overline{y}_n \end{bmatrix} \qquad (13)$$

where:

$s_{i,j} = \underline{a}_{i,j} \quad s_{i+n,j+n} = \overline{a}_{i,j}$, $s_{i,j+n} = 0$, $s_{i+n,j} = 0$, where $i = 1,...,n \quad j = 1,...,n$.

The structure of *S* is $S = \begin{bmatrix} \underline{B} & 0 \\ 0 & \overline{B} \end{bmatrix}$, it follows that *S* is non-singular if $|\underline{B}| \neq 0$ and $|\overline{B}| \neq 0$.

S^{-1} has the form:

$$S^{-1} = [t_{i,j}] = \begin{bmatrix} \underline{B}^{-1} & 0 \\ 0 & \overline{B}^{-1} \end{bmatrix}$$

and the solution is:

$$\begin{bmatrix} \underline{x}_i \\ \overline{x}_i \end{bmatrix} = \begin{bmatrix} \underline{B}^{-1} & 0 \\ 0 & \overline{B}^{-1} \end{bmatrix} \begin{bmatrix} \underline{y}_i \\ \overline{y}_i \end{bmatrix}$$

$$\underline{x}_i = \underline{B}^{-1} \underline{y}_i$$

$$\overline{x}_i = \overline{B}^{-1} \overline{y}_i$$

Note that in this case the solution can be found by solving separately the sub-system for $\underline{x}$ and $\overline{x}$ (as a consequence, this solution interval is thus included in the one proposed in section 3).

If ex-post $\underline{x} \geq \overline{x}$, we define the fuzzy solution $u = \{ \underline{u}_i, \overline{u}_i, i = 1,...,n \}$ as:

$$\underline{u}_i = \min \{ \underline{x}_i, \overline{x}_i \}$$

$$\overline{u}_i = \max \{ \underline{x}_i, \overline{x}_i \}$$

We say that there is no solution if the peak value falls outside of the support of the fuzzy number.

The solution is a strong fuzzy solution ($\underline{x}_i \leq \overline{x}_i$) iff: $\sum_{j=1}^{n} (t_{n+i,n+j} \overline{y}_j - t_{i,j} \underline{y}_j) \geq 0$. Otherwise we have a weak fuzzy solution, i.e. a solution where $\underline{x}_i \geq \overline{x}_i$.

Proof.

$$\underline{x}_i = \sum_{j=1}^{n} t_{ij} \underline{y}_j$$

$$\overline{x}_i = \sum_{j=1}^{n} t_{n+i,n+j} \overline{y}_j$$

$$\overline{x}_i - \underline{x}_i = \sum_{j=1}^{n} t_{n+i,n+j} \overline{y}_j - \sum_{j=1}^{n} t_{ij} \underline{y}_j .$$

In the particular case in which y is a crisp vector, the solution x_i is a strong fuzzy solution iff: $\sum_{j=1}^{n} (t_{n+i,n+j} - t_{i,j}) y_j \geq 0$.

Next we have to investigate all the other systems that arise from a shift to the right hand side or the left hand side of any term of each equation. The final solution for each unknown is investigated by taking the minimum and the maximum of the different solutions found in each system. If $x_1(1) \notin [\underline{x}_1(0), \overline{x}_1(0)]$ or $x_2(1) \notin [\underline{x}_2(0), \overline{x}_2(0)]$, then we conclude that there is no solution to the system. We present the whole methodology in the financial example in section 6.

6. The financial problem

In this section we investigate the solution to a problem that arises in the derivation of the artificial probabilities in a lattice framework.

Let us consider a one period model where $t \in \{0,1\}$ is time, at time one we have two different states of the world, one where the market is bullish and one where the market is bearish and we have two financial instruments, the money market account and the stock. The money market account is worth *1* at time zero and *1+r* at time one, where *r* is the risk free interest rate. The stock price is worth *S* at time zero and at time 1 it is worth *Su* in state "market bullish" and *Sd* in state "market bearish", where *u* and *d* are the two jump factors, that satisfy the following no arbitrage condition:

$d < (1+r) < u$.

We are looking for the so called artificial probabilities that ensures that the price of an instrument at time 0 is equal to the expected value, discounted at the risk free rate, of the payoff of the instrument at time 1.

Writing this equality for each instrument we get:

$$\begin{cases} \dfrac{(1+r)}{(1+r)} p_u + \dfrac{(1+r)}{(1+r)} p_d = 1 \\ \dfrac{Sd}{(1+r)} p_d + \dfrac{Su}{(1+r)} p_u = S \end{cases}$$

The system can be simplified as follows:

$$\begin{cases} p_u + p_d = 1 \\ dp_d + up_u = (1+r) \end{cases}$$

where p_u and p_d are the unknown artificial probabilities of an up and a down move respectively; *u* and *d* are the up and down jump factors of the stock and *(1+r)* is the end of the period value of one unit of money invested at the risk free rate.

The crisp solution to the system is:

$$\begin{cases} p_u = \dfrac{(1+r)-d}{u-d} \\ p_d = \dfrac{u-(1+r)}{u-d} \end{cases}$$

There is no fuzziness in the risk free rate, since it is given at time zero, while it is usually difficult to precisely estimate the up and down jump factors of the stock, as pointed out in [6]. Thus it is convenient to represent the two jump factors with fuzzy numbers, in particular we can use the two triangular fuzzy numbers $u=(u_1, u_2, u_3)$ and $d=(d_1, d_2, d_3)$, as in [6].

The fuzzified system is the following:

$$\begin{cases} p_u + p_d = 1 \\ [\underline{d},\overline{d}]p_d + [\underline{u},\overline{u}]p_u = (1+r) \end{cases}$$

where $\underline{k} = k_1 + \alpha(k_2 - k_1)$ and $\overline{k} = k_3 - \alpha(k_3 - k_2)$ for $k = d, u$.

Let us investigate the three solutions proposed by Buckley et al. (2002).

In order to find X_J, we investigate:

$$\begin{cases} p_u + p_d \le 1 \\ p_u + p_d \ge 1 \\ \underline{u}p_u + \underline{d}p_d \le (1+r) \\ \overline{u}p_u + \overline{d}p_d \ge (1+r) \end{cases}$$

for p_u and p_d in the first quadrant (see Figure 1).

The solution is:

$$\begin{cases} p_{Ju} = [\underline{p}_{Ju}, \overline{p}_{Ju}] = \left[\dfrac{(1+r)-\overline{d}}{\overline{u}-\overline{d}}, \dfrac{(1+r)-\underline{d}}{\underline{u}-\underline{d}} \right] \\ p_{Jd} = [\underline{p}_{Jd}, \overline{p}_{Jd}] = \left[\dfrac{\underline{u}-(1+r)}{\underline{u}-\underline{d}}, \dfrac{\overline{u}-(1+r)}{\overline{u}-\overline{d}} \right] \end{cases}$$

In order to find X_E, we fuzzify the crisp solution using the extension principle:

$$\begin{cases} p_{Eu} = [\underline{p}_{Eu}, \overline{p}_{Eu}] = \left[\dfrac{(1+r)-\overline{d}}{\overline{u}-\overline{d}}, \dfrac{(1+r)-\underline{d}}{\underline{u}-\underline{d}} \right] \\ p_{Ed} = [\underline{p}_{Ed}, \overline{p}_{Ed}] = \left[\dfrac{\underline{u}-(1+r)}{\underline{u}-\underline{d}}, \dfrac{\overline{u}-(1+r)}{\overline{u}-\overline{d}} \right] \end{cases}$$

In order to find X_I, we fuzzify the crisp solution using fuzzy arithmetic:

$$\begin{cases} p_{Iu} = [\underline{p}_{Iu}, \overline{p}_{Iu}] = \left[\dfrac{(1+r)-\overline{d}}{\overline{u}-\underline{d}}, \dfrac{(1+r)-\underline{d}}{\underline{u}-\overline{d}} \right] \\ p_{Id} = [\underline{p}_{Id}, \overline{p}_{Id}] = \left[\dfrac{\underline{u}-(1+r)}{\overline{u}-\underline{d}}, \dfrac{\overline{u}-(1+r)}{\underline{u}-\overline{d}} \right] \end{cases}$$

Note that in this case we loose the property that a fuzzy system is an extension of a crisp system: e.g. by inspection of $\overline{p}_{Iu}$, it contains both the terms $\overline{d}$ and $\underline{d}$, as a consequence there can be no crisp system (except for $\alpha = 1$) with crisp values for *u* and *d* that lead to a crisp value of $\overline{p}_{Iu}$.

Moreover it is easy to check that X_I contains X_E, thus in this case $X_J = X_E \leq X_I$. As X_J exists, we take X_J as the Buckley et al. (2002) solution.

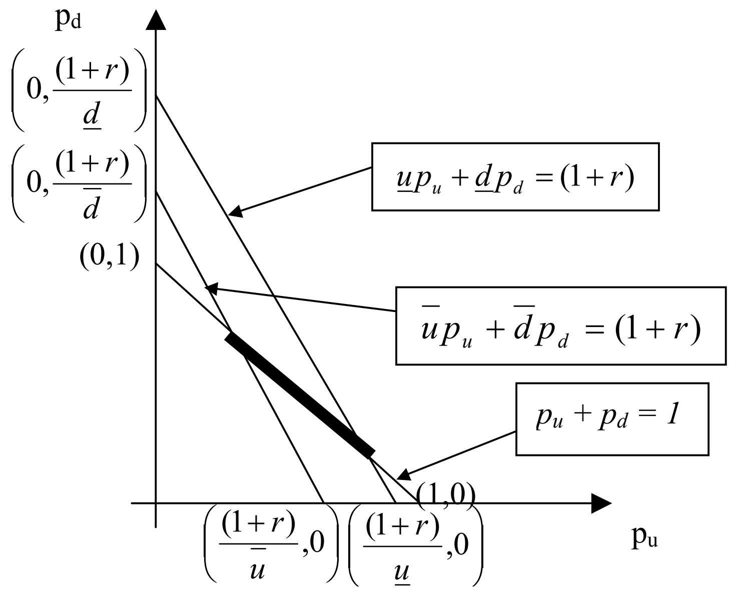

Figure 1. The solution X_J (in bold).

The solution proposed in section 3 is obtained by solving the following systems:

$$\begin{cases} p_u + p_d = 1 \\ \underline{d}p_d + \underline{u}p_u = (1+r) \end{cases} \tag{14}$$

$$\begin{cases} p_u + p_d = 1 \\ \underline{d}p_d + \overline{u}p_u = (1+r) \end{cases} \tag{15}$$

$$\begin{cases} p_u + p_d = 1 \\ \overline{d}p_d + \underline{u}p_u = (1+r) \end{cases} \tag{16}$$

$$\begin{cases} p_u + p_d = 1 \\ \overline{d}p_d + \overline{u}p_u = (1+r) \end{cases} \tag{17}$$

The solution to system 14 is:

$$\begin{cases} p_u = \dfrac{(1+r) - \underline{d}}{\underline{u} - \underline{d}} \\[2ex] p_d = \dfrac{\underline{u} - (1+r)}{\underline{u} - \underline{d}} \end{cases}$$

The solution to system 15 is:

$$\begin{cases} p_u = \dfrac{(1+r)-\underline{d}}{\overline{u}-\underline{d}} \\ p_d = \dfrac{\overline{u}-(1+r)}{\overline{u}-\underline{d}} \end{cases}$$

The solution to system 16 is:

$$\begin{cases} p_u = \dfrac{(1+r)-\overline{d}}{\underline{u}-\overline{d}} \\ p_d = \dfrac{\underline{u}-(1+r)}{\underline{u}-\overline{d}} \end{cases}$$

The solution to system 17 is:

$$\begin{cases} p_u = \dfrac{(1+r)-\overline{d}}{\overline{u}-\overline{d}} \\ p_d = \dfrac{\overline{u}-(1+r)}{\overline{u}-\overline{d}} \end{cases}$$

The final solution is investigated by taking the minimum and the maximum for each unknown over the set of possible solutions:

$$\underline{p}_u = \min(\underline{p}_u(system14), \underline{p}_u(system15), \underline{p}_u(system16), \underline{p}_u(system17))$$
$$\overline{p}_u = \max(\overline{p}_u(system14), \overline{p}_u(system15), \overline{p}_u(system16), \overline{p}_u(system17))$$
$$\underline{p}_d = \min(\underline{p}_d(system14), \underline{p}_d(system15), \underline{p}_d(system16), \underline{p}_d(system17))$$
$$\overline{p}_d = \max(\overline{p}_d(system14), \overline{p}_d(system15), \overline{p}_d(system16), \overline{p}_d(system17))$$

we get:

$$\begin{cases} p_u = [\underline{p}_u, \overline{p}_u] = \left[\dfrac{(1+r)-\overline{d}}{\overline{u}-\overline{d}}, \dfrac{(1+r)-\underline{d}}{\underline{u}-\underline{d}}\right] \\ p_d = [\underline{p}_d, \overline{p}_d] = \left[\dfrac{\underline{u}-(1+r)}{\underline{u}-\underline{d}}, \dfrac{\overline{u}-(1+r)}{\overline{u}-\overline{d}}\right] \end{cases}$$

The solution proposed in section 5 is obtained as follows.

Depending on how we write the left hand side and the right hand side of the system we have four different fuzzy systems and thus we can find four different solutions.

Solution 1) As u and d are fuzzy numbers then the fuzzy system is:

$$\begin{cases} [\underline{p_u}, \overline{p_u}] + [\underline{p_d}, \overline{p_d}] = [1,1] \\ [\underline{d}, \overline{d}][\underline{p_d}, \overline{p_d}] + [\underline{u}, \overline{u}][\underline{p_u}, \overline{p_u}] = [(1+r),(1+r)] \end{cases}$$

by applying the rules of operations between fuzzy numbers, and keeping in mind that we are looking for probabilities, i.e. positive numbers[2], we get:

$$
\begin{cases}
\underline{p}_u + \underline{p}_d = 1 \\
\underline{d}\,\underline{p}_d + \underline{u}\,\underline{p}_u = (1+r) \\
\overline{p}_u + \overline{p}_d = 1 \\
\overline{d}\,\overline{p}_d + \overline{u}\,\overline{p}_u = (1+r)
\end{cases}
\tag{18}
$$

or, written in matrix form:

$$
S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ \underline{d} & \underline{u} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \overline{d} & \overline{u} \end{bmatrix}, \qquad x = \begin{bmatrix} \underline{p}_d \\ \underline{p}_u \\ \overline{p}_d \\ \overline{p}_u \end{bmatrix}, \qquad y = \begin{bmatrix} 1 \\ (1+r) \\ 1 \\ (1+r) \end{bmatrix}
$$

which leads to the solution (which is the one proposed in Muzzioli and Torricelli, 2001):

$$
\begin{cases}
\underline{p}_u = \dfrac{(1+r) - \underline{d}}{\underline{u} - \underline{d}} \\
\underline{p}_d = \dfrac{\underline{u} - (1+r)}{\underline{u} - \underline{d}} \\
\overline{p}_u = \dfrac{(1+r) - \overline{d}}{\overline{u} - \overline{d}} \\
\overline{p}_d = \dfrac{\overline{u} - (1+r)}{\overline{u} - \overline{d}}
\end{cases}
$$

we can observe that $\overline{p}_u < \underline{p}_u$, that is we have a weak fuzzy solution, as defined in [3]. Thus, we define the solution: $\underline{p}_u^1 = \overline{p}_u$ and $\overline{p}_u^1 = \underline{p}_u$ and the final solution is:

$$
\begin{cases}
p_u^1 = [\underline{p}_u^1, \overline{p}_u^1] = \left[\dfrac{(1+r) - \overline{d}}{\overline{u} - \overline{d}}, \dfrac{(1+r) - \underline{d}}{\underline{u} - \underline{d}} \right] \\
p_d^1 = [\underline{p}_d^1, \overline{p}_d^1] = \left[\dfrac{\underline{u} - (1+r)}{\underline{u} - \underline{d}}, \dfrac{\overline{u} - (1+r)}{\overline{u} - \overline{d}} \right]
\end{cases}
$$

Solution 2) The crisp system can be rewritten as:

$$
\begin{cases}
p_u = 1 - p_d \\
dp_d + up_u = (1+r)
\end{cases}
$$

applying the rules of fuzzy numbers we get:

---

[2] By the no arbitrage condition the possibility that a probability is equal to zero is ruled out.

$$\begin{cases} \underline{p_u} = 1 - \overline{p_d} \\ \underline{d}\,\underline{p_d} + \underline{u}\,\underline{p_u} = (1+r) \\ \overline{p_u} = 1 - \underline{p_d} \\ \overline{d}\,\overline{p_d} + \overline{u}\,\overline{p_u} = (1+r) \end{cases}$$

which leads to the solution (which is the one proposed in [6]):

$$\begin{cases} \underline{p_u} = \dfrac{(1+r)(\overline{u}+\underline{d}) - \underline{d}(\overline{d}+\overline{u})}{\underline{u}\overline{u} - \underline{d}\overline{d}} \\ \underline{p_d} = 1 - \overline{p_u} \\ \overline{p_u} = \dfrac{(1+r)(\underline{u}+\overline{d}) - \overline{d}(\underline{u}+\underline{d})}{\underline{u}\overline{u} - \underline{d}\overline{d}} \\ \overline{p_d} = 1 - \underline{p_u} \end{cases}$$

it can be shown that $\overline{p_u} < \underline{p_u}$ and $\overline{p_d} < \underline{p_d}$, i.e. a weak fuzzy solution. Let us denote the interval of solutions: $\underline{p}_u^2 = \overline{p}_u$ and $\overline{p}_u^2 = \underline{p_u}$, the interval of solution is:

$$\begin{cases} p_u^2 = [\underline{p}_u^2, \overline{p}_u^2] = \left[\dfrac{(1+r)(\underline{u}+\overline{d}) - \overline{d}(\underline{u}+\underline{d})}{\underline{u}\overline{u} - \underline{d}\overline{d}}, \dfrac{(1+r)(\overline{u}+\underline{d}) - \underline{d}(\overline{d}+\overline{u})}{\underline{u}\overline{u} - \underline{d}\overline{d}} \right] \\ p_d^2 = [\underline{p}_d^2, \overline{p}_d^2] = \left[1 - \dfrac{(1+r)(\overline{u}+\underline{d}) - \underline{d}(\overline{d}+\overline{u})}{\underline{u}\overline{u} - \underline{d}\overline{d}}, 1 - \dfrac{(1+r)(\underline{u}+\overline{d}) - \overline{d}(\underline{u}+\underline{d})}{\underline{u}\overline{u} - \underline{d}\overline{d}} \right] \end{cases}$$

It can be shown that $\underline{p}_u^2 > \underline{p}_u^1$ and $\overline{p}_u^1 > \overline{p}_u^2$, i.e. that this solution interval is contained in the solution interval of system 18.

Solution 3) If we rewrite the crisp system as:

$$\begin{cases} p_u = 1 - p_d \\ dp_d = (1+r) - up_u \end{cases}$$

applying the rules of fuzzy numbers we get:

$$\begin{cases} \underline{p_u} = 1 - \overline{p_d} \\ \underline{d}\,\underline{p_d} = (1+r) - \overline{u}\,\overline{p_u} \\ \overline{p_u} = 1 - \underline{p_d} \\ \overline{d}\,\overline{p_d} = (1+r) - \underline{u}\,\underline{p_u} \end{cases}$$

and the solution is:

$$\begin{cases} \underline{p}_u = \dfrac{(1+r)-\overline{d}}{\underline{u}-\overline{d}} \\ \underline{p}_d = 1-\overline{p}_u \\ \overline{p}_u = \dfrac{(1+r)-\underline{d}}{\overline{u}-\underline{d}} \\ \overline{p}_d = 1-\underline{p}_u \end{cases}$$

Also in this case we have a weak fuzzy solution, since it can be shown that it may happen, depending on the values of u and d that or $\overline{p}_u < \underline{p}_u$ and $\overline{p}_d < \underline{p}_d$, or $\overline{p}_u > \underline{p}_u$ and $\overline{p}_d > \underline{p}_d$.

Moreover, it can be shown that also in this case the solution interval is contained in the solution interval of system 18.

Solution 4) If we rewrite the crisp system as:

$$\begin{cases} p_u + p_d = 1 \\ dp_d = (1+r) - up_u \end{cases}$$

applying the rules of fuzzy numbers we get:

$$\begin{cases} \underline{p}_u + \underline{p}_d = 1 \\ \underline{d}\,\underline{p}_d = (1+r) - \overline{u}\,\overline{p}_u \\ \overline{p}_u + \overline{p}_d = 1 \\ \overline{d}\,\overline{p}_d = (1+r) - \underline{u}\,\underline{p}_u \end{cases}$$

the solution is:

$$\begin{cases} \underline{p}_u = \dfrac{(1+r)(\overline{u}+\overline{d})-\overline{d}(\underline{d}+\overline{u})}{\underline{u}\overline{u}-\underline{d}\overline{d}} \\ \underline{p}_d = 1-\underline{p}_u \\ \overline{p}_u = \dfrac{(1+r)(\underline{u}+\underline{d})-\underline{d}(\underline{u}+\overline{d})}{\underline{u}\overline{u}-\underline{d}\overline{d}} \\ \overline{p}_d = 1-\overline{p}_u \end{cases}$$

it may happen, depending on the values of u and d that $\overline{p}_u < \underline{p}_u$ and $\overline{p}_d > \underline{p}_d$, or $\overline{p}_u > \underline{p}_u$ and $\overline{p}_d < \underline{p}_d$. Moreover, it can be shown that also in this case, the interval for p_u is contained in the solution interval of system 18.

In sum, we take as solution the solution interval of system 18:

$$\begin{cases} p_u = [\underline{p}_u, \overline{p}_u] = \left[\dfrac{(1+r)-\overline{d}}{\overline{u}-\overline{d}}, \dfrac{(1+r)-\underline{d}}{\underline{u}-\underline{d}} \right] \\ p_d = [\underline{p}_d, \overline{p}_d] = \left[\dfrac{\underline{u}-(1+r)}{\underline{u}-\underline{d}}, \dfrac{\overline{u}-(1+r)}{\overline{u}-\overline{d}} \right] \end{cases}$$

for two main reasons: first it is the widest and contains all the artificial probabilities consistent with the given up and down jump factors, second it is the only one that represents a fuzzy number vector, as it is formally shown, by analysing the behaviour of p_u and p_d, in [6].

7. Conclusions

In this paper we have analysed the solution of a fuzzy system when the classical solution based on standard fuzzy mathematics fails to exist. In particular we have considered the solution of the system Ax=b with A squared matrix with positive fuzzy coefficients and y crisp vector of positive elements. We have proposed two different solutions that are based on the work of Buckley et al. (2002) and Friedman, Ming and Kandel (1998), and we have applied them to the financial problem of finding the artificial probabilities in a lattice framework. The two solutions proposed fulfill two properties: they contain all the other possible solutions and they are represented by a fuzzy vector.

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Appendix 1.

Let $X=[\underline{x}_j,\overline{x}_j]$, $Y=[\underline{y}_j,\overline{y}_j]$, the rules for interval addition and subtraction are [5]:

$X+Y=[\underline{x}+\underline{y},\ \ \overline{x}+\overline{y}]$

$X-Y=[\underline{x}-\overline{y},\ \ \overline{x}-\underline{y}]$

The rules to evaluate the product between the two fuzzy numbers $a_{i,j}$ and x_j are the following [5]:

1) $\underline{a}_{i,j}\geq 0$, $\underline{x}_j\geq 0$ $\underline{a_{i,j}x_j}=\underline{a}_{i,j}\underline{x}_j$, $\overline{a_{i,j}x_j}=\overline{a}_{i,j}\overline{x}_j$

2) $\underline{a}_{i,j}\geq 0$, $\underline{x}_j<0<\overline{x}_j$ $\underline{a_{i,j}x_j}=\overline{a}_{i,j}\underline{x}_j$, $\overline{a_{i,j}x_j}=\overline{a}_{i,j}\overline{x}_j$

3) $\underline{a}_{i,j}\geq 0$, $\overline{x}_j\leq 0$ $\underline{a_{i,j}x_j}=\overline{a}_{i,j}\underline{x}_j$, $\overline{a_{i,j}x_j}=\underline{a}_{i,j}\overline{x}_j$

4) $\overline{a}_{i,j}\leq 0$, $\underline{x}_j\geq 0$ $\underline{a_{i,j}x_j}=\underline{a}_{i,j}\overline{x}_j$, $\overline{a_{i,j}x_j}=\overline{a}_{i,j}\underline{x}_j$

5) $\overline{a}_{i,j}\leq 0$, $\underline{x}_j<0<\overline{x}_j$ $\underline{a_{i,j}x_j}=\underline{a}_{i,j}\overline{x}_j$, $\overline{a_{i,j}x_j}=\underline{a}_{i,j}\underline{x}_j$

6) $\overline{a}_{i,j}\leq 0$, $\overline{x}_j\leq 0$ $\underline{a_{i,j}x_j}=\overline{a}_{i,j}\overline{x}_j$, $\overline{a_{i,j}x_j}=\underline{a}_{i,j}\underline{x}_j$

7) $\underline{a}_{i,j}<0<\overline{a}_{i,j}$, $\underline{x}_j\geq 0$ $\underline{a_{i,j}x_j}=\underline{a}_{i,j}\overline{x}_j$, $\overline{a_{i,j}x_j}=\overline{a}_{i,j}\overline{x}_j$

8) $\underline{a}_{i,j}<0<\overline{a}_{i,j}$, $\underline{x}_j<0<\overline{x}_j$ $\underline{a_{i,j}x_j}=\min(\underline{a}_{i,j}\overline{x}_j,\overline{a}_{i,j}\underline{x}_j)$ $\overline{a_{i,j}x_j}=\max(\underline{a}_{i,j}\underline{x}_j,\overline{a}_{i,j}\overline{x}_j)$

9) $\underline{a}_{i,j}<0<\overline{a}_{i,j}$, $\overline{x}_j\leq 0$ $\underline{a_{i,j}x_j}=\overline{a}_{i,j}\underline{x}_j$, $\overline{a_{i,j}x_j}=\underline{a}_{i,j}\underline{x}_j$

If $\underline{a}_{i,j}\geq 0$, then $\underline{a_{i,j}x_j}=\min(\underline{a}_{i,j}\underline{x}_j,\overline{a}_{i,j}\underline{x}_j)$ and $\overline{a_{i,j}x_j}=\max(\overline{a}_{i,j}\overline{x}_j,\underline{a}_{i,j}\overline{x}_j)$

If $\underline{a}_{i,j}\leq 0$, then $\underline{a_{i,j}x_j}=\min(\underline{a}_{i,j}\overline{x}_j,\overline{a}_{i,j}\overline{x}_j)$ and $\overline{a_{i,j}x_j}=\max(\overline{a}_{i,j}\underline{x}_j,\underline{a}_{i,j}\underline{x}_j)$

If $\underline{a}_{i,j}<0<\overline{a}_{i,j}$, then $\underline{a_{i,j}x_j}=\min(\underline{a}_{i,j}\overline{x}_j,\overline{a}_{i,j}\underline{x}_j)$ and $\overline{a_{i,j}x_j}=\max(\overline{a}_{i,j}\overline{x}_j,\underline{a}_{i,j}\overline{x}_j)$

if $a_{i,j}$ is crisp then cases 7), 8) and 9) are impossible;

cases 1) 2) and 3) reduce to:

1’) $a_{i,j} \geq 0,$ $\underline{a_{i,j}x_j} = a_{i,j}\underline{x}_j,$ $\overline{a_{i,j}x_j} = a_{i,j}\overline{x}_j;$

cases 4) 5) and 6) reduce to:

2’) $a_{i,j} < 0,$ $\underline{a_{i,j}x_j} = a_{i,j}\overline{x}_j,$ $\overline{a_{i,j}x_j} = a_{i,j}\underline{x}_j.$

Appendix 2. Examples.

1. Consider the system:

$$\begin{cases} 2x_1 - 3x_2 = [-3+4r, 5-4r] \\ -4x_1 + x_2 = [-10+11r, 3-2r] \end{cases}$$

The solution proposed in section 3 is obtained by solving the following four systems and taking the minimum and the maximum for each alpha for each unknown:

$$\begin{cases} 2x_1 - 3x_2 = -3+4r \\ -4x_1 + x_2 = -10+11r \end{cases}$$

$$\begin{cases} x_1 = 3.3 - 3.7\alpha \\ x_2 = 3.2 - 3.8\alpha \end{cases}$$

$$\begin{cases} 2x_1 - 3x_2 = 5-4r \\ -4x_1 + x_2 = 3-2r \end{cases}$$

$$\begin{cases} x_1 = -1.4 + \alpha \\ x_2 = -2.6 + 2\alpha \end{cases}$$

$$\begin{cases} 2x_1 - 3x_2 = -3+4r \\ -4x_1 + x_2 = 3-2r \end{cases}$$

$$\begin{cases} x_1 = -0.6 + 0.2\alpha \\ x_2 = 0.6 - 1.2\alpha \end{cases}$$

$$\begin{cases} 2x_1 - 3x_2 = 5-4r \\ -4x_1 + x_2 = -10+11r \end{cases}$$

$$\begin{cases} x_1 = 2.5 - 2.9\alpha \\ x_2 = -0.6\alpha, \end{cases}$$

The final solution is:

$$\begin{cases} x_1 = [-1.4+\alpha,\ 3.3-3.7\alpha] = (-1.4, -0.4, 3.3) \\ x_2 = [-2.6+2\alpha,\ 3.2-3.8\alpha] = (-2.6, -0.6, 3.2) \end{cases}$$

Note that in this case, in which only the vector *y* if fuzzy, the same solution is obtained by using the simple method proposed at the end of section 3, i.e. finding at $\alpha=0$ and at $\alpha=1$ the solutions to the four systems, taking the min and the max for each unknown and imposing ex-post the triangular shape.

The Friedman, Ming and Kandel (1998) solution is:

$$\begin{cases} x_1 = [-0.6+0.2\alpha,\ 2.5-2.9\alpha] = (-0.6,-0.4,2.5) \\ x_2 = [-0.6\alpha, 0.6-1.2\alpha] = (0,-0.6,\ 0.6) \end{cases}$$

Note that as x_2 has the peak value that is outside the support of the fuzzy number, the final solution is a weak fuzzy solution:

$$\begin{cases} x_1 = [-0.6+0.2\alpha,\ 2.5-2.9\alpha] = (-0.6,-0.4,2.5) \\ x_2 = [-0.6, 0.6-1.2\alpha] = (-0.6,-0.6,\ 0.6) \end{cases}$$

2. Consider the system:

$$\begin{cases} 2x_1 - 4x_2 = [-3+5r, 4-2r] \\ 2x_1 + 5x_2 = [-10+4r, -4-2r] \end{cases}$$

The solution proposed in section 3 is the following:

$$\begin{cases} x_1 = [-3.0556+2.2778\alpha, 0.2222-\alpha] = (-3.0556,\ -0.7778,\ 0.2222) \\ x_2 = [-1.5556+0.6667\alpha, -0.1111-0.7778\alpha] = (-1.5556,\ -0.8889,\ -0.1111) \end{cases}$$

The Friedman, Ming and Kandel (1998) solution is the following:

$$\begin{cases} x_1 = [-4.1667, -0.7778, 1.3333] \\ x_2 = [-0.3333, -0.8889,\ -1.3333] \end{cases}$$

Note that in this case it is not contained in the solution interval proposed in section 3.

Appendix 3. Examples of the financial problem.

1. Consider the system:

$$\begin{cases} [1,1]x_1 + [1,1]x_2 = [1,1] \\ [0.5+\alpha(0.8-0.5),\ 0.85-\alpha(0.85-0.8)]x_1 + [1.18+\alpha(1.25-1.18),\ 2-\alpha(2-1.25)]x_2 = [1.1\ ,1.1] \end{cases}$$

where $d = (0.5, 0.8, 0.85)$, $u = (1.18, 1.25, 2)$ and $r=0.1$.

The final solution is:

$$\begin{cases} p_u = \left[\dfrac{1.1-(0.85-\alpha(0.85-0.8))}{(2-\alpha(2-1.25))-(0.85-\alpha(0.85-0.8))}, \dfrac{1.1-(0.5+\alpha(0.8-0.5))}{(1.18+\alpha(1.25-1.18))-((0.5+\alpha(0.8-0.5))} \right] \\ p_d = \left[\dfrac{(1.18+\alpha(1.25-1.18))-1.1}{(1.18+\alpha(1.25-1.18))-((0.5+\alpha(0.8-0.5))}, \dfrac{(2-\alpha(2-1.25))-1.1}{(2-\alpha(2-1.25))-(0.85-\alpha(0.85-0.8))} \right] \end{cases}$$

The four solutions to the system are graphed in Figure 1. Note that solution 1 is the widest.

2. Consider the system:

$$\begin{cases} [1,1]x_1 + [1,1]x_2 = [1,1] \\ [0.7+\alpha(0.8-0.7),\ 0.85-\alpha(0.85-0.8)]x_1 + [1.2+\alpha(1.55-1.20),\ 1.60-\alpha(1.60-1.55)]x_2 = [1.05,1.05] \end{cases}$$

The four solutions to the system are graphed in Figure 2. Note that solution 1 is the widest and it is the only one that represents a fuzzy vector.

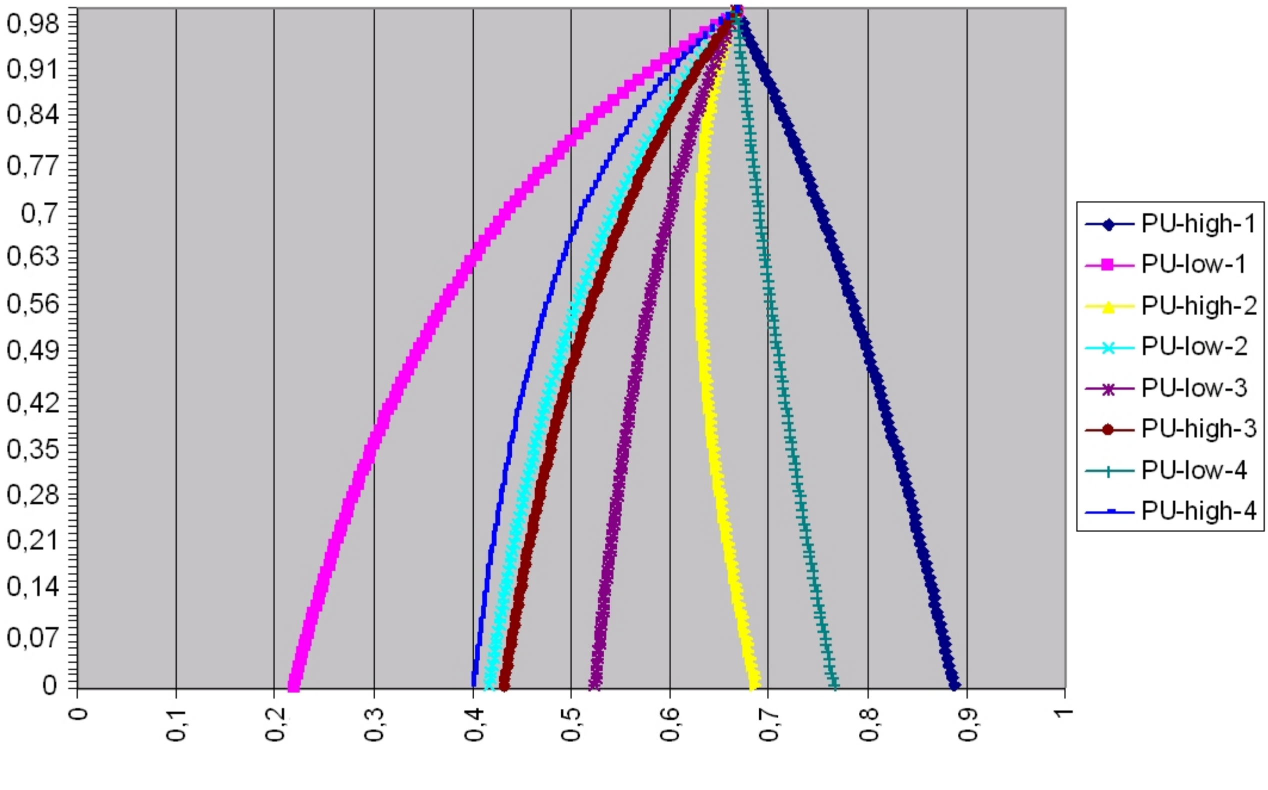

Figure 2. The graph of the four solutions to example 1.

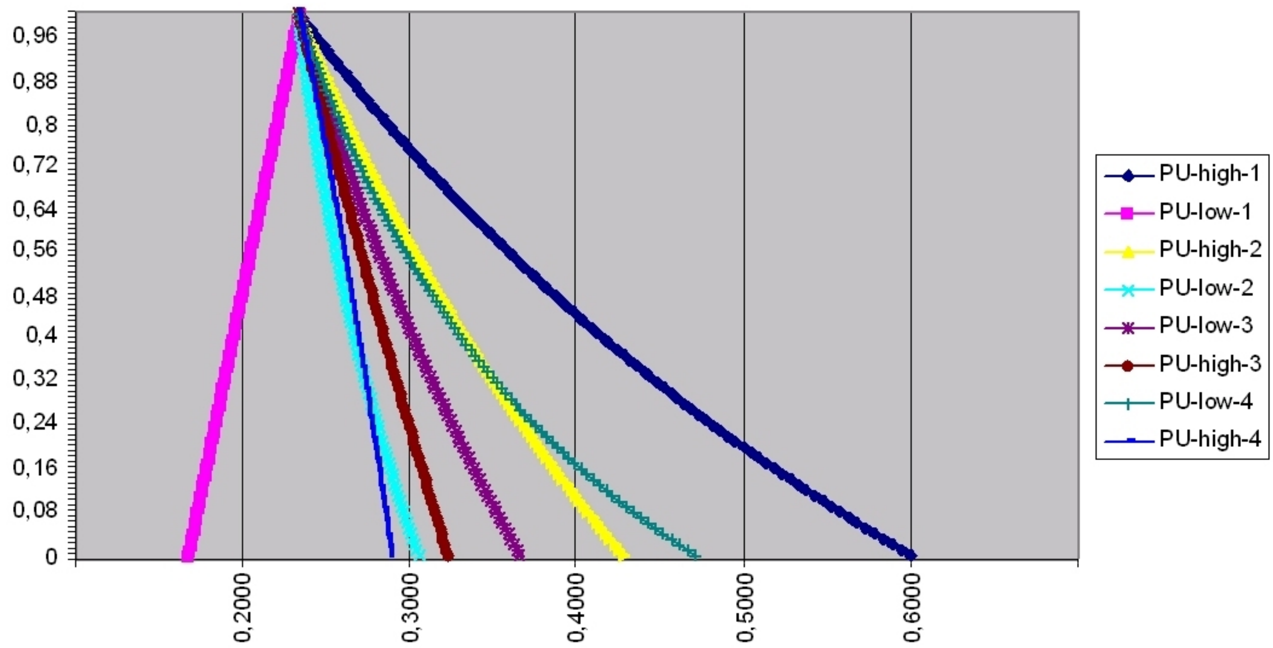

Figure 3. The graph of the four solutions to example 2.