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The no Arbitrage Condition in Option Implied Trees: Evidence from the Italian Index Options Market

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THE NO ARBITRAGE CONDITION IN OPTION IMPLIED TREES:
EVIDENCE FROM THE ITALIAN INDEX OPTIONS MARKET

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Abstract

A major issue in the construction of implied trees is the no arbitrage property preservation. Within the literature on deterministic smile-consistent trees using forward induction, two major contributions are: Derman and Kani (1994) and Barle and Cakici (1998). The former proposes a methodology to override the nodes that violate the no arbitrage condition. The latter extends the Derman and Kani's algorithm, in order to increase its stability in the presence of high interest rates.

The aim of the present paper is to modify the Derman and Kani's methodology in order to improve the fit of the implied tree to option prices. The proposed methodology is compared with Barle and Cakici both in the sample and out of sample with Italian index options data. Overall findings support a better performance of the modified Derman and Kani's methodology.

Keywords: Binomial tree, implied volatility, calibration.

JEL classification: G13, G14.

1. Introduction

After the October 1987 crash, option markets exhibited implied volatilities that varied across different strikes (smile effect) and different times to expiration (term structure of the volatility), in contrast with the Black and Scholes assumption of constant volatility.

In order to capture the implied volatility dependence on strike and time to maturity, different smile-consistent no-arbitrage models have been proposed in the literature, which can be classified either as deterministic or stochastic volatility models[1]. Deterministic volatility models (see e.g. Derman and Kani (1994), Barle and Cakici (1998), Rubinstein (1994), Jackwerth (1997), Dupire (1994)) derive endogenously from European option prices the instantaneous volatility as a deterministic function of the asset price and time. Stochastic volatility models (see e.g. Derman and Kani (1997), Britten-Jones and Neuberger (2000), Ledoit and Santa Clara (1998)) allow for a no-arbitrage evolution of the implied volatility surface.

Deterministic volatility models have both theoretical and practical advantages: they preserve the no-arbitrage pricing property of the Black and Scholes model and are easily implementable. With the exception of Dupire (1994), which is developed in continuous time, most models are developed in discrete time. Among the latter, some (Derman and Kani (1994), Barle and Cakici (1998), Li (2001)) use forward induction in the derivation of the implied trees, others (Rubinstein (1994), Jackwerth (1997)), use backward induction[2]. The Rubinstein (1994) model is based on the assumption that different paths that lead to the same ending node have the same risk neutral probability, it captures only the smile effect and it is not useful for pricing path dependent options. The Jackwerth (1997) model, extend Rubinstein's by allowing the implied tree to fit intermediate maturity options, thus capturing both the smile effect and the term structure of the volatility. The main advantages of deriving implied trees by forward induction is that only observable data are used and, in contrast to backward induction, no estimation of ending risk neutral probabilities is needed.

A few papers empirically test the pricing performance of deterministic smile-consistent option pricing models (see among others, Dumas et al. (1998), Lim and Zhi (2002), Brandt and Wu (2002), Hull and Suo (2002), Linaras and Skiadopoulos (2005)), while, as underlined by Linaras and Skiadopoulos (2005), stochastic volatility smile-consistent

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[1] See Bates (2003) for a survey on the approaches taken in option pricing and Skiadopoulos (2001) for a taxonomy and an extensive survey on smile-consistent no arbitrage models.

[2] This paper departs here from the terminology used by Skiadopoulos (2001) in that forward induction models are meant as those that use also forward induction and backward induction ones are those that use only backward induction.

models have not been tested yet, because of various computational limitations. The empirical tests compare different types of smile-consistent deterministic models w.r.t constant volatility models (such as Black and Scholes (1973), Cox-Ross-Rubinstein (1979)). The evidence on the pricing performance of deterministic smile-consistent models is mixed. Dumas et al. (1998) and Brandt and Wu (2002) find that they do not perform better than an ad hoc procedure that smoothes Black and Scholes (1973) implied volatilities across strikes and time to expiration. By contrast, Hull and Suo (2002) find that they are superior to Black and Scholes in the pricing of exotic options. In Lim and Zhi (2002) and Linaras and Skiadopoulos (2005), the pricing performance of different types of deterministic-smile consistent models is shown to strongly depend on various factors (option class chosen, moneyness and time to expiration). No apparent superiority of one specific model w.r.t. the others emerges.

A major issue that negatively affects the pricing performance of implied trees based on forward induction is the occurrence of negative probabilities, which following Linaras and Skiadopoulos (2005) can be addressed to as "bad probabilities". Negative probabilities indicate the presence of arbitrage opportunities. Derman and Kani (1994) propose a methodology to override the nodes that violate the no arbitrage condition. Nonetheless negative probabilities are frequently found, questioning the correct replication of the observed smile. Barle and Cakici (1998) extend the Derman and Kani's algorithm, in order to increase its stability, in particular in the presence of high interest rates. Negative probabilities turn out less frequently, but in the presence of increasing interest rates and smile slopes, the fit to the smile is poor. In order to solve the problem, Li (2001) proposes to derive implied trees, by assuming constant nodal probabilities equal to 0.5. However, the Li's model strongly hinges on the assumption that the risk neutral measure exists. Moreover it is not appropriate for pricing path-dependent options, since all paths leading to the same node are equally likely (as in the Rubinstein's model).

In sum, focusing on deterministic volatility models based on forward induction, Derman and Kani (1994) remains comparatively the most suitable. In order to remove the problem of negative probabilities, the aim of this paper is to propose a modification of the no arbitrage test used to this purpose. The proposed methodology is a modified Derman and Kani model (MDK from now on) and will be compared with the Barle and Cakici (1998) implied tree, both in the sample and out of sample. The empirical validation of the different implied trees is performed by using a data set, Italian index options over the

period March 2000 - December 2003, which to our knowledge has not been yet used to the same purpose.

This paper is organized as follows. Section 2 recalls the basics of the Derman and Kani implied tree and Section 3 briefly illustrates the Barle and Cakici extension. Section 4 describes the methodology proposed in this paper in order to override nodes that violate the no arbitrage condition. Section 5 describes the data set and the implementation. The last Section concludes.

2. The Derman and Kani implied tree

Derman and Kani (1994) (DK) construct an implied tree using forward induction. Let j=0,…,n be the number of levels of the tree, that are spaced by Δt. As the tree recombines, i=1,...,j+1 is the number of nodes at level j. Forward induction is used to compute level j variables given level j-1 variables as inputs. The initial inputs are the riskless interest rate, the stock price at time zero and the smile function. The latter is used to determine the price of the appropriate ATM call and put prices.

DK methodology assumes that the tree has been implied out to level j-1. Figure 1 focuses on levels j-1 and j. The known stock price $S_{i,j-1}$, can evolve into $S_{i+1,j}$ in state up and $S_{i,j}$, in state down. The risk neutral probability of an up jump is $p_{i,j}$. The Arrow-Debreu price, $\lambda_{i,j}$, is computed by forward induction as the sum over all paths leading to node (i,j) of the product of the risk neutral probabilities discounted at the risk-free rate at each node in each path.

[Figure 1 about here]

The problem is how to imply nodes at level j. There are 2j+1 unknowns: j+1 stock prices ($S_{i,j}$) and j risk neutral probabilities of an up move ($p_{i,j}$) , (see Figure 1). Hence, 2j+1 equations are needed: the first 2j equations require the theoretical value of j forwards and j options expiring at time j to match their market values (for the upper part of the tree call options are used, while for the lower part of the tree, put options), the remaining degree of freedom is used to require the tree to develop around the current stock price (centring condition). The centring condition is given by equation (1) if the level is even and by equation (2) if the level is odd:

$$S_{\frac{j}{2}+1} = S_{0,0} \tag{1}$$

$$S_{\frac{j+3}{2}}S_{\frac{j+1}{2}} = S_{0,0}{}^2 \tag{2}$$

For the upper part of the tree the recursive equation to compute $S_{i+1,j}$ given $S_{i,j}$ is:

4

$$S_{i+1,j} = \frac{S_{i,j}\left[e^{r\Delta t}C_{i,j-1} - \sum\right] - I_{i,j-1}S_{i,j-1}\left(F_{i,j-1} - S_{i,j}\right)}{\left[e^{r\Delta t}C_{i,j-1} - \sum\right] - I_{i,j-1}\left(F_{i,j-1} - S_{i,j}\right)} \tag{3}$$

where $\sum = \sum_{k=i+1}^{j} I_{k,j-1}\left(F_{k,j-1} - S_{i,j-1}\right)$, $F_{i,j-1}$ is the forward value of $S_{i,j-1}$ and $C_{i,j-1}$ is the price

at time 0 of a call with strike $S_{i,j-1}$ and maturity j. It is computed using a j step tree with constant volatility obtained from the smile function.

In order to use equation (3), an initial node $S_{i,j}$ is needed. If the number of nodes is even, the central node is chosen to be equal to the current spot (equation (1)); if the number of nodes is odd, combining equations (2) and (3) yields the following equation:

$$S_{i+1,j} = \frac{S_{0,0}\left[e^{r\Delta t}C_{0,j-1} + I_{i,j-1}S_{0,0} - \sum\right]}{I_{i,j-1}F_{i,j-1} - e^{r\Delta t}C_{0,j-1} + \sum} \tag{4}$$

For nodes in the lower part of the tree, a put with strike $S_{i,j-1}$ instead of a call, is used.

The recursive formula that provides $S_{i,j}$ given $S_{i+1,j}$ is obtained:

$$S_{i,j} = \frac{S_{i+1,j}\left[e^{r\Delta t}P_{i,j-1} - \sum{}'\right] + I_{i,j-1}S_{i,j-1}\left(F_{i,j-1} - S_{i+1,j}\right)}{\left[e^{r\Delta t}P_{i,j-1} - \sum{}'\right] + I_{i,j-1}\left(F_{i,j-1} - S_{i+1,j}\right)} \tag{5}$$

where $\sum{}' = \sum_{k=1}^{i-1} I_{k,j-1}\left(S_{i,j-1} - F_{k,j-1}\right)$ and $P_{i,j-1}$ is the price at time 0 of a call with strike $S_{i,j-1}$

and maturity j. It is computed using a j step tree with constant volatility obtained from the smile function. By repeating this process at each level, the entire tree is generated (see Figure 2).

The artificial probabilities of each node must belong to]0,1[and violation of this condition implies the presence of riskless arbitrage opportunities. In fact, if $p_{i,j} \geq 1$ then $S_{i+1,j} \leq F_i$, if $p_{i,j} \leq 0$ then $S_{i,j} \geq F_i$.

Thus, at each iteration, the following condition is tested:

$$F_{i,j-1} < S_{i,j} < F_{i-1,j-1} \tag{6}$$

Were this not verified, DK override the stock price $S_{i,j}$ as follows:

$$\ln(S_{i,j}) - \ln(S_{i-1,j}) = \ln(S_{i,j-1}) - \ln(S_{i-1,j-1}) \tag{7}$$

Condition (7) keeps the logarithmic spacing of stock prices in nodes i and i-1 at level j, the same as in the corresponding nodes at the previous level j-1.

[Figure 2 about here]

3. The Barle and Cakici modification

Barle and Cakici (1998) (BC) note that the DK model fails to accurately reproduce the smile, because negative transition probabilities are frequently found. In order to ensure that the artificial probabilities remain in the interval]0,1[, they propose three major modifications, which are essentially based on the use of the forward price. First, they do not fix the centre of the tree equal to today stock price, but they let it evolve at the risk free rate. Second, they choose the options' strike equal to the forward $F_{i,j}$. Third, if a stock price violates the no-arbitrage condition, they choose to override it by setting:

$$S_{i,j} = \frac{F_{i-1,j-1} + F_{i,j-1}}{2} \tag{8}$$

Even though BC modifications do help in avoiding negative transition probabilities, in the presence of increasing interest rates and smile slope, BC model still fails to accurately reproduce the smile.

4. The modified Derman and Kani

This section illustrates the modifications to the Derman and Kani methodology proposed in this paper in order to avoid arbitrage opportunities.

As stressed by BC, equation (7) does not guarantee that the stock price satisfies the no-arbitrage condition (6). Therefore DK method allow negative probabilities and hence results in a poor replication of traded option prices. The BC solution to the problem is to take the average of forward values $F_{i-1,j-1}$ and $F_{i,j-1}$ (equation (8)).

However, equation (8), which rules out arbitrage opportunities in BC model since the centre of the tree increases at the risk free rate, does not eliminate arbitrage opportunities in the DK model, where the centre of the tree is constrained to remain equal to the initial stock price. The no-arbitrage test in the DK model, that is based on the comparison of node at level j ($S_{i,j}$) with the nodes at level j-1 ($S_{i-1,j-1}$ and $S_{i,j-1}$)), has to be integrated by a condition that takes into account the relation of the forward value of the newly determined stock price $S_{i,j}$, with respect to the center of the tree at level j+1, that is fixed and known in advance.

In order to investigate which is the condition to be fulfilled by the newly determined stock price, it is necessary to distinguish different cases (see Figure 3) depending on both:

a) nodes being in the upper (yellow nodes) or lower (red nodes) part of the tree or on the boundary (blue nodes on the upper boundary and orange nodes on the lower boundary)

b) the relation between the dividend yield, d, and the risk-free rate, r.

The following Subsection investigates the nodes in the upper part of the tree, Subsection 4.2 the nodes in the lower part and Subsection 4.3 the nodes on the boundary.

[Figure 3 about here]

4.1 Nodes in the upper part.

Figure 4 illustrates four nodes, $S_{i,j}$, where $i=1,...n+1$ indicate the node and $j=0,..,n$ the level of the tree, along with the forward values $F_{i,j}$ in the case $r > d$. In this case $F_{i,j}$ is strictly bigger than $S_{i,j}$ for each $i=1,...n+1$, $j=0,..,n$. In particular, for node $S_{i,j}$ the forward $F_{i,j}$ is bigger than $S_{0,0}$, that is fixed and the no arbitrage relation that $S_{i,j}$ has to fulfil is:

$$F_{i-1,j-1} = S_{i-1,j-1}e^{(r-\partial)\Delta t} < S_{i,j} < S_{i,j-1} * e^{(r-\partial)\Delta t} = F_{i,j-1} \tag{9}$$

i.e. $S_{i,j}$ must lie between the forward values of $S_{i-1,j-1}$ and $S_{i,j-1}$.

If $S_{i,j}$ violates the no-arbitrage condition, equation (8) is used in order to impose a value for $S_{i,j}$, consistent to the no-arbitrage condition.

The same happens if $r = d$, as illustrated in Figure 5.

[Figures 4, 5 and 6 about here]

Figure 6 illustrates the case $r < d$. In this case the forward $F_{i,j}$, being lower than $S_{i,j}$, can also be lower than $S_{i,j+1}=S_{0,0}$, causing a no-arbitrage violation. Therefore, the no-arbitrage condition (6) has to be restricted as follows:

$$S_{i-1,j-1} / e^{(r-d)\Delta t} < S_{i,j} < S_{i,j-1}e^{(r-d)\Delta t} = F_{i,j-1} \tag{10}$$

If node $S_{i,j}$ violates the no-arbitrage condition (10), it is determined by means of the following:

$$S_{i,j} = \frac{S_{i-1,j-1} / e^{(r-d)\Delta t} + F_{i,j-1}}{2}$$

Condition (10) obviously guarantees no-arbitrage opportunities when $S_{i-1,j-1}=S_{0,0}$, since $S_{i,j+1}=S_{i-1,j-1}=S_{0,0}$. In the general case, $S_{i,j}$ is any node in the upper part of the tree (not necessarily the one above the center) e.g., node $S_{i,j-1}$. In order to have no arbitrage opportunities, since $F_{i,j-1} = S_{i,j-1}e^{(r-d)\Delta t} > S_{i,j}$ and $F_{i,j} = S_{i,j}e^{(r-d)\Delta t} > S_{0,0}$, it is necessary that

$$S_{i,j-1} > S_{0,0} / e^{2(r-d)\Delta t}.$$

The first inequality of the no arbitrage condition (10) requires $S_{i,j-1} > S_{i-1,j-2} / e^{(r-\partial)\Delta t}$, since $S_{i-1,j-2} > S_{0,0} / e^{(r-\partial)\Delta t}$, then $S_{i,j-1} > S_{0,0} / e^{2(r-d)\Delta t}$.

4.2 Nodes in the lower part of the tree

Figure 7 illustrate the nodes in the lower part of the tree when $r > d$. This case mirrors the case for nodes in the upper part of the tree when $r < d$.

[Figure 7 about here]

In this case, in order to avoid arbitrage opportunities, $F_{i,j} = S_{i,j} e^{(r-d)\Delta t}$, the forward value of $S_{i,j}$, is constrained to be less than $S_{0,0}$, that is fixed.

This implies, in turn, the same condition for all the other nodes in the lower part of the tree, e.g., $S_{i-1,j-1}$, should be less than $S_{0,0} / e^{2(r-d)\Delta t}$ since $F_{i-1,j-1} = S_{i-1,j-1} e^{(r-d)\Delta t} < S_{i,j}$ and $F_{i,j} = S_{i,j} e^{(r-d)\Delta t} < S_{0,0}$. By the same argument, $S_{i-2,j-2} < S_{0,0} / e^{3(r-d)\Delta t}$.

Therefore, in order to avoid arbitrage opportunities it is necessary for $S_{i,j}$ to satisfy the following no-arbitrage condition :

$$F_{i-1,j-1} = S_{i-1,j-1} e^{(r-d)\Delta t} < S_{i,j} < S_{i,j-1} / e^{(r-d)\Delta t} \tag{11}$$

If $S_{i,j}$ violates the no-arbitrage condition, it is obtained by the following:

$$S_{i,j} = \frac{S_{i,j-1} / e^{(r-d)\Delta t} + F_{i-1,j-1}}{2}$$

Condition (11) guarantees no-arbitrage opportunities: e.g. for node $S_{i-1,j-1}$, the last inequality of the no arbitrage condition (11) requires: $S_{i-1,j-1} < S_{i-1,j-2} / e^{(r-\partial)\Delta t}$, since $S_{i-1,j-2} < S_{0,0} / e^{(r-\partial)\Delta t}$ then $S_{i-1,j-1} < S_{0,0} / e^{2(r-d)\Delta t}$.

If $r \leq d$ (Figures 8 and 9) then $F_{i,j}$ is less than or equal to $S_{i,j}$ for each i=1,...n+1, j=0,..,n. Therefore the forward $F_{i,j}$ is also less than or equal to $S_{0,0}$, that is fixed and the no arbitrage relation (9) is still valid. If $S_{i,j}$ violates the no-arbitrage condition, equation (8) is used in order to substitute it.

[Figures 8 and 9 about here]

4.3 Nodes at the boundary of the tree.

In order to introduce a no arbitrage test for the nodes at the boundary of the tree, nodes in the upper part have to be examined separately form nodes in the lower part of the tree. For nodes in the upper part, the no-arbitrage condition is:

$$
\begin{aligned}
r \geq d \quad & F_{i-1,j-1} = S_{i-1,j-1} e^{(r-d)\Delta t} < S_{i,j} \\
r < d \quad & S_{i-1,j-1} / e^{(r-d)\Delta t} < S_{i,j}
\end{aligned}
\tag{12}
$$

Figure 10 illustrates the case $r < d$.

If a node violates condition (12), it is obtained by:

$$
S_{i,j} = \begin{cases}
S_{i-1,j-1} e^{2(r-d)\Delta t} & \text{if} \quad r > d \\
S_{i-1,j-1} e^{r\Delta t} & \text{if} \quad r = d \\
S_{i-1,j-1} / e^{2(r-d)\Delta t} & \text{if} \quad r < d
\end{cases}
\tag{13}
$$

[Figures 10 and 11 about here]

For nodes in the lower part, the no-arbitrage condition is:

$$
\begin{aligned}
r > d \quad & S_{i,j} < S_{i,j-1} / e^{(r-d)\Delta t} \\
r \leq d \quad & S_{i,j} < S_{i,j-1} e^{(r-d)\Delta t}
\end{aligned}
\tag{14}
$$

Figure 11 illustrates the case $r > d$.

If a node violates condition (14) it is obtained by:

$$
S_{i,j} = \begin{cases}
S_{i,j-1} / e^{2(r-d)\Delta t} & \text{if} \quad r > d \\
S_{i,j-1} / e^{r\Delta t} & \text{if} \quad r = d \\
S_{i,j-1} e^{2(r-d)\Delta t} & \text{if} \quad r < d
\end{cases}
\tag{15}
$$

5. An application

In this section the modified Derman and Kani (MDK) and the BC implied trees are implemented both in the sample and out of sample, by using the MIB30 index options data set covering the period March 2000 - December 2003.

5.1 The data set

The data set consists of closing prices of Mib30-index options, with maturities up to one year, recorded from 20 March 2000 to 19 December 2003. The option contracts on the Mib30 index (MibO) were introduced in the Italian Derivatives Market (IDEM) in November 1995. Mib30 index options are European options on the Mib30 index, which is a capital weighted

index composed of 30 major stocks quoted on the Italian market. It is adjusted for stocks splits, changes in capital and for extraordinary dividends, but not for ordinary dividends[3]. Mib30 options quoted in index points, representing a value of 2.5 €, with six different expirations (four quarterly -March, June, September and December- and two monthly -the nearest two months). The last trading day is the third Friday of the expiry month.

The underlying is the Mib30-index recorded in the same time period. As the Mib30 is not adjusted for dividends, the daily dividend yield, that is available in Bloomberg, is used in order to compute the appropriate value for the index, as follows:

$$S_{t+\Delta t} = S_t e^{-d_t \Delta t}$$

where S_t is the Mib30 value at time t, δ_t is the dividend yield and Δt is the time increment.

As a proxy for the risk-free rate the Euribor rates with maturities up to one year are used. Yields to maturity are computed by linear interpolation. The whole data-set source is Bloomberg.

Two different filters are applied to the data set. First, options with less than two days and more than one year to maturity are excluded. Second, trading dates with less than 11 options traded are left out.

5.2 The methodology

The methodology develops into three steps: first a smile function has to be estimated, second, the two implied trees, the MDK and the BC are derived, third the two methods are compared both in the sample and out of sample.

In order to estimate the smile function, a linear function of the form:

$$s(X) = a_0 + a_1 X$$

is used, where the Black and Scholes implied volatilities are computed by using the bisection method in C++. The function s depend only on the strike price of the option, therefore, in order to estimate[4] the parameters options are grouped in classes with the same trading day and time to maturity.

As the choice of the number (odd or even) of the binomial tree levels implies different estimates of the price, the prices are computed as the average between odd and even levels. A binomial tree with 25 and 26 levels is assumed (as e.g. in Barle and Cakici (1998)).

The MDK implied tree is derived following the procedure illustrated in Derman and Kani

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[3] As from September 2004, the derivatives on the S&P/Mib index have been replacing those on the Mib30 index on the Italian Derivatives Market (IDEM). However, the features of the options on the S&P/Mib are very similar to those on the Mib30. For more details see www.borsaitalia.it

[4] The parameters are estimated each day by solving a least square problem implemented in GAMS ver. 20.7, using the solver MINOS 5.4.

(1994), with the only exception that the no arbitrage condition detailed in Section 4 is used to exclude arbitrage violations. The BC implied tree is derived following the procedure detailed in Barle and Cakici (1998).

The two methods are compared both in and out of sample. In the sample, the binomial tree is implied from option prices (following the DK or BC methodology) and the same set of options is priced on the tree. Out of sample, date t+1 theoretical prices are computed on date t implied trees and compared with t+1 market prices.

In order to gauge the pricing performance of the two methods three indicators that are computed on each day for each option class and then averaged across the sample are used: the mean squared error (MSE), the mean squared relative error (MSRE) and the index of mispricing (MISP); they are respectively defined as follows:

$$MSE = \frac{1}{m} \sum_{i=1}^{m} (P_i^T - P_i^M)^2$$

$$MSRE = \frac{1}{m} \sum_{i=1}^{m} \left(\frac{P_i^T - P_i^M}{P_i^M} \right)^2$$

$$MISP = \frac{\sum_{i=1}^{m} \left(\dfrac{P_i^T - P_i^M}{P_i^M} \right)}{\sum_{i=1}^{m} \left| \dfrac{P_i^T - P_i^M}{P_i^M} \right|}$$

P_i^T and P_i^M are respectively the theoretical and the market price of option i, $i = 1, \ldots, m$ and m is the number of options in the class.

The MSE is an indicator of the implied tree fit to option prices, it naturally increases with the moneyness of the option. The MSRE is a percentage error and is usually higher for out of the money options. The mispricing index ranges from –1 to 1 and indicates, on average, the overpricing (positive MISP) or underpricing (negative MISP) induced by the method.

In order to detect which options classes are best priced by each model, options are divided according to their moneyness. Five classes, according to the indicator of moneyness $M = S/(Ke^{-rT})$, where S is the underlying value and K the strike price of the option, are individuated (DOM (deep-out-of-the-money, call options: $M < 0.9$ and put options: $M > 1.1$), OM (out-of-the-money, call options: $0.9 \leq M < 0.98$ and put options: $1.02 < M \leq 1.1$), AM (at-the-money, call and put options: $0.98 \leq M \leq 1.02$), IM (in-the-money, call options: $1.02 < M \leq 1.1$ and put options $0.9 \leq M < 0.98$), DIM (deep-in-the-money, call options: $M > 1.1$ and put options: $M < 0.9$)).

5.3 An empirical comparison

Table 1 reports the in the sample performance. The MDK performs better than the BC implied tree according to all the indicators. They both underprice on average, but the underpricing of the MDK is substantially lower. In particular, the better performance of the MDK can be attributed to the better pricing of put options. Table 2 shows the performance with respect to moneyness. The better performance of the MDK can be attributed to the better pricing of deep out of the money and out of the money options. In particular, deep out of the money put are the best priced by the MDK.

Table 3 reports the out of sample performance, which, according to the MSE and the MSRE is for both models worse than the in the sample one, the only exception being the MISP index. Overall, the three indicators jointly point to a better pricing performance of the MDK w.r.t. the BC. Similarly to the in the sample analysis, the better performance is mainly due to a better pricing of put options. Table 4 shows the performance with respect to moneyness and results confirm the better pricing performance of the MDK as for out of the money put options.

Table 5 illustrates the number of no arbitrage violations detected, that required a replacement of the stock price. The minimum and the maximum number of replacements is higher for the MDK, because the model implies a test for no arbitrage also at the boundary nodes (not present in the BC). But according to the total and average number of replacements, the MDK implied tree is superior, in that it encounters a fewer arbitrage violations. Therefore the MDK better fits the smile and this turns out in a better pricing performance.

6. Conclusions

This paper has proposed a modification of the Derman and Kani no-arbitrage test in order to improve the fit to option prices. The no arbitrage condition has been examined by including dividends into the picture. In order to improve the fit to deep out of the money options, a no-arbitrage test for the nodes at the boundary of the tree has been introduced. The modified Derman and Kani and the Barle and Cakici implied trees have been compared, both in the sample and out of sample by using the MIB30 index options data set covering the period March 2000 - December 2003.

The empirical results suggest that the modified Derman and Kani performs better than the Barle and Cakici, both in the sample and out of sample. In particular, the better performance of the modified Derman and Kani can be attributed to the better pricing of out of the money

put options, i.e. a better fit in the lower part of the tree. The better fit can be also explained by a lower number of no arbitrage violations for the modified Derman and Kani. This results in a lower number of stock price replacements and therefore in a better fit to traded option prices.

REFERENCES

Barle S., N. Cakici, 1998. How to grow a smiling tree. *Journal of Financial Engineering,* 7 (2) 127-146.

Bates D. S., 2003. Empiriical option pricing: a retrospection. *Journal of Econometrics*, 116, 387-404.

Black F., Scholes M., 1973. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81, 637,654.

Brandt M. W., Wu T., 2002. Cross sectional tests of deterministic volatility functions. *Journal of Empirical Finance*, 9, 525-550.

Britten-Jones M., Neuberger A., 2000. Option prices, implied price processes and stochastic volatility. *Journal of Finance*, 55 (2) 839-866 .

Cox J., Ross S., Rubinstein M., 1979. Option pricing: a simplified approach. *Journal of Financial Economics*, 7, 229-263.

Derman E., Kani I., 1994. Riding on a smile. *Risk*, 7 (2) 32-39.

Derman E., Kani I., 1997. Stochastic implied trees: arbitrage pricing with stochastic term and strike structure of volatility. *International Journal of Theoretical and Applied Finance*, 1, 61-110.

Dumas B., Fleming J., Whaley R.E., 1998. Implied volatility functions: empirical tests. *Journal of Finance*, 53, 2059-2106.

Dupire B., 1994. Pricing with a smile. *Risk*, 7 (1) 18-20.

Hull J.C., Suo W., 2002. A methodology for assessing model risk and its applications to the implied volatility function model. *Journal of Financial and Quantitative Analysis*, 37, 297-318.

Jackwerth J., 1997. Generalized binomial trees. *Journal of Derivatives*, 5, 7-17.

Ledoit O. Santa Clara P., 1998. Relative pricing of options with stochastic volatility. *Working Paper of the University of California, Los Angeles.*

Li Y., 2001. A new algorithm for constructing implied binomial trees: does the implied model fit any volatility smile? *Journal of Computational Finance*, 4 (2) 69-95.

Lim K., Zhi D., 2002. Pricing options using implied trees: evidence from FTSE-100 options. *Journal of Futures Markets*, 22, 601-626.

Rubinstein M., 1994. Implied binomial trees. *Journal of Finance* 49 (3) 771-818.

Skiadopoulos G., 2001. Volatility Smile Consistent Option Models: A Survey. *International Journal of Theoretical and Applied Finance*, 4 (3) 403-437.

Linaras H., Skiadopoulos G., 2005. Implied Volatility Trees and Pricing Performance: Evidence from the S&P100 Options, PP05-146 Working Paper, Financial Options Research Centre, University of Warwick.

	MSE	MSRE	MISP	MSE Call	MSRE Call	MISP Call	MSE Put	MSRE Put	MISP Put
BC	66990.09	0.251537	-0.15105	67867.52	0.211266	-0.21894	66112.67	0.291807	-0.14986
MDK	62471.18	0.153191	-0.10046	67982.50	0.160461	-0.15916	56959.86	0.14592	-0.13433

Table 1- In the sample pricing performance of Barle and Cakici (BC) and modified Derman and Kani (MDK)

	MSE	MSRE	MISP	MSE Call	MSRE Call	MISP Call	MSE Put	MSRE Put	MISP Put
				Panel A: Barle and Cakici					
DOM	35642.76	0.3037854	-0.470175	28075.04	0.359441	-0.48799	43210.48	0.24813	-0.443005
OM	22203.76	0.7359008	0.3222684	19131.57	0.4473269	0.3928	25275.94	1.024475	0.2389769
AM	53518.95	0.1756063	0.325635	44915.07	0.0897105	0.37256	62122.82	0.261502	0.2758833
IM	88252.18	0.0119939	0.1733406	69241.42	0.0116981	0.17496	107262.94	0.01229	0.1716602
DIM	104571.75	0.043355	-0.149994	111912.04	0.0533579	-0.48673	97231.47	0.033352	0.2879822
				Panel B: Modified Derman and Kani					
DOM	16004.08	0.292285	-0.42987	27824.8	0.331302	-0.43979	4183.35	0.253268	-0.41572
OM	22358.19	0.289524	0.364623	19781.66	0.215145	0.452649	24934.72	0.363904	0.2472
AM	53184.01	0.047235	0.347771	46471.31	0.056083	0.402992	59896.7	0.038388	0.286506
IM	87808.08	0.012079	0.179577	70198.48	0.011748	0.182918	105417.7	0.012411	0.176097
DIM	102876.3	0.04533	-0.13928	111327.2	0.055987	-0.47762	94425.33	0.034674	0.300215

Table 2- In the sample pricing performance w.r.t. moneyness classes

	MSE	MSRE	MISP	MSE Call	MSRE Call	MISP Call	MSE Put	MSRE Put	MISP Put
BC	86541.08	0.85473	-0.0911	91956.33	0.368451	-0.1307	81125.83	1.341015	-0.1029
MDK	81413.18	0.814915	-0.0416	90898.25	0.304321	-0.07193	71928.11	1.325508	-0.0851

Table 3- Out of sample pricing performance of Barle and Cakici (BC) and modified Derman and Kani (MDK)

	MSE	MSRE	MISP	MSE Call	MSRE Call	MISP Call	MSE Put	MSRE Put	MISP Put
				Panel A: Barle and Cakici					
DOM	57515.20	2.062914	-0.37318	53023.47	1.2007393	-0.403661	62006.93	2.9250891	-0.33661
OM	55526.54	0.645279	0.30333	62072.65	0.523179	0.3784889	48980.44	0.7673798	0.206086
AM	82273.18	0.334826	0.329003	82202.22	0.1023297	0.3916961	82344.13	0.5673231	0.260514
IM	107891.83	0.01353	0.18001	95436.59	0.0136097	0.1881692	120347.07	0.0134498	0.17108
DIM	112659.27	0.049648	-0.14624	121991.44	0.0601006	-0.479763	103327.11	0.0391963	0.296142
				Panel B: Modified Derman and Kani					
DOM	36908.97	2.18845	-0.33172	52359.08	1.156266	-0.34492	21458.86	3.220633	-0.31668
OM	55305.11	0.37555	0.334418	62191.81	0.334923	0.418155	48418.42	0.416176	0.217824
AM	81713.41	0.05317	0.353098	83196.65	0.062505	0.415864	80230.17	0.043835	0.280123
IM	107189.6	0.013712	0.186893	95389.55	0.013662	0.195527	118989.7	0.013761	0.177404
DIM	110847.8	0.053293	-0.1349	119228.4	0.063996	-0.47384	102467.2	0.042589	0.307908

Table 4- Out of sample pricing performance w.r.t. moneyness classes

	BC	MDK
mean	57.56056	47.60131
min	0	15
max	239	287
tot.	257123	204162

Table 5- No arbitrage violations for Barle and Cakici (BC) and modified Derman and

Kani (MDK)
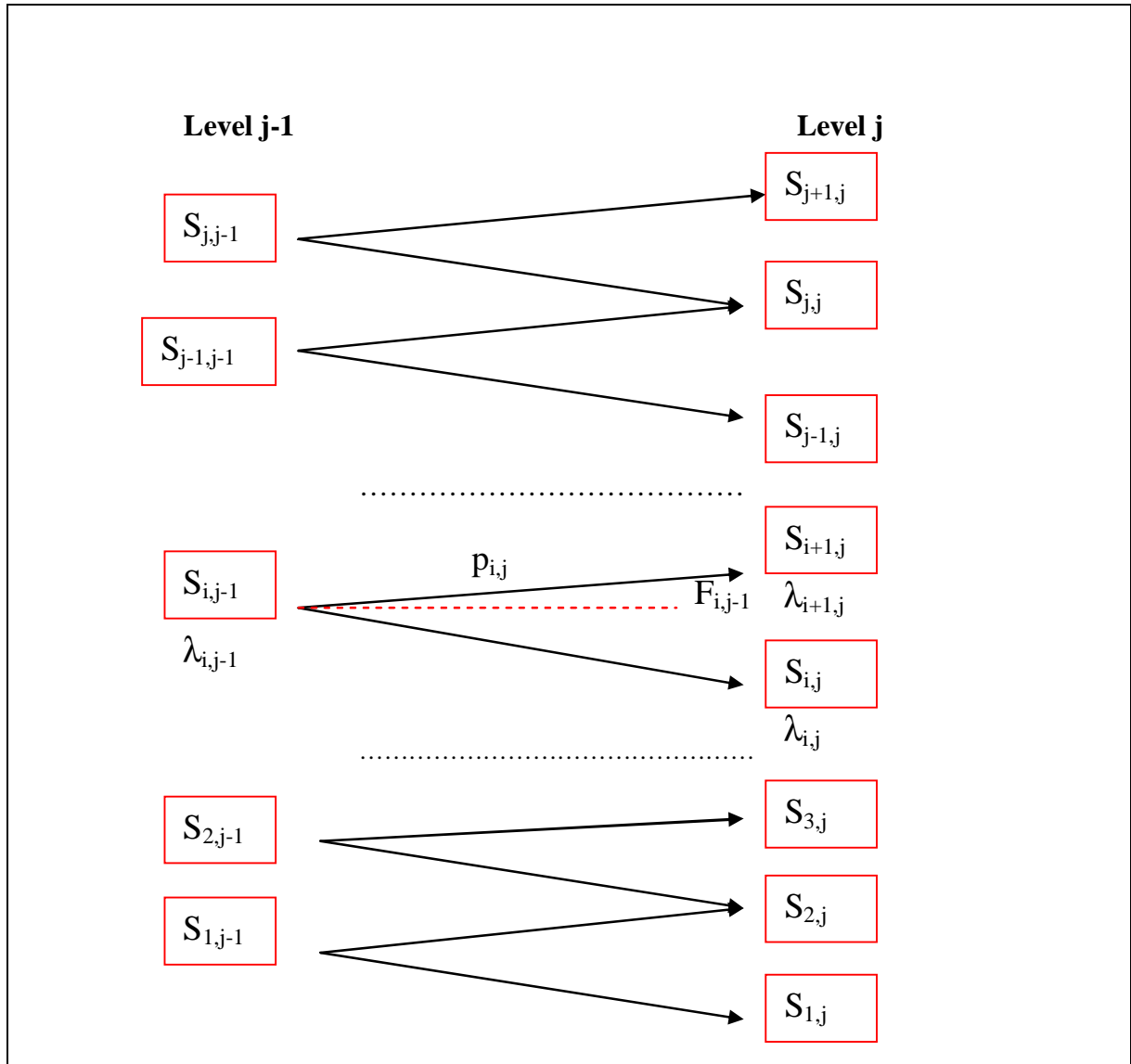
Figure 1. Levels j-1 and j of the tree.
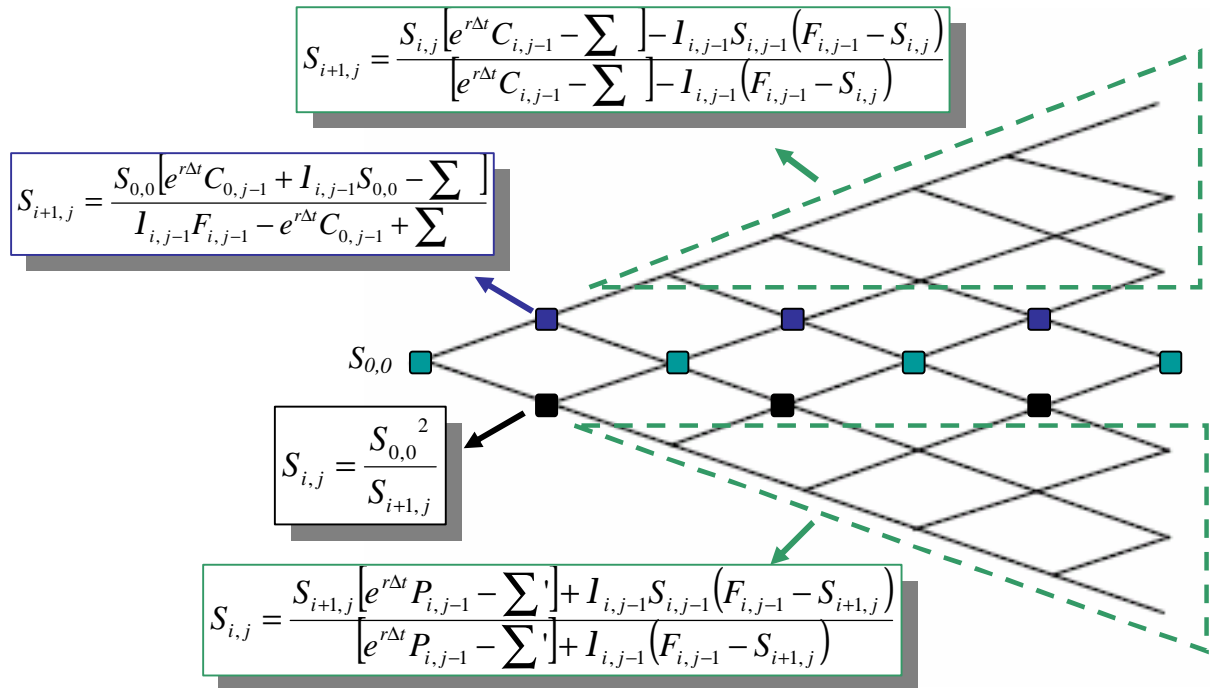
$$S_{i+1,j} = \frac{S_{i,j}\left[e^{r\Delta t}C_{i,j-1} - \sum\right] - I_{i,j-1}S_{i,j-1}\left(F_{i,j-1} - S_{i,j}\right)}{\left[e^{r\Delta t}C_{i,j-1} - \sum\right] - I_{i,j-1}\left(F_{i,j-1} - S_{i,j}\right)}$$

$$S_{i+1,j} = \frac{S_{0,0}\left[e^{r\Delta t}C_{0,j-1} + I_{i,j-1}S_{0,0} - \sum\right]}{I_{i,j-1}F_{i,j-1} - e^{r\Delta t}C_{0,j-1} + \sum}$$

$$S_{i,j} = \frac{S_{0,0}^{\;2}}{S_{i+1,j}}$$

$$S_{i,j} = \frac{S_{i+1,j}\left[e^{r\Delta t}P_{i,j-1} - \sum{}'\right] + I_{i,j-1}S_{i,j-1}\left(F_{i,j-1} - S_{i+1,j}\right)}{\left[e^{r\Delta t}P_{i,j-1} - \sum{}'\right] + I_{i,j-1}\left(F_{i,j-1} - S_{i+1,j}\right)}$$

Figure 2. The equations in the Derman and Kani tree.

$$S_{i,j} = \begin{cases} S_{i,j} = \dfrac{S_{i-1,j-1}/e^{(r-d)\Delta t} + F_{i,j-1}}{2} & if \quad r < d \\[3mm] S_{i,j} = \dfrac{F_{i,j-1} + F_{i-1,j-1}}{2} & if \quad r \ge d \end{cases}$$

$$S_{i,j} = \begin{cases} S_{i-1,j-1}e^{2(r-d)\Delta t} & if \quad r > d \\ S_{i-1,j-1}e^{r\Delta t} & if \quad r = d \\ S_{i-1,j-1}/e^{2(r-d)\Delta t} & if \quad r < d \end{cases}$$

$$S_{i,j} = \begin{cases} S_{i,j} = \dfrac{S_{i,j-1}/e^{(r-d)\Delta t} + F_{i-1,j-1}}{2} & if \quad r > d \\[3mm] S_{i,j} = \dfrac{F_{i,j-1} + F_{i-1,j-1}}{2} & if \quad r \le d \end{cases}$$

$$S_{i,j} = \begin{cases} S_{i,j-1}/e^{2(r-d)\Delta t} & if \quad r > d \\ S_{i,j-1}/e^{r\Delta t} & if \quad r = d \\ S_{i,j-1}e^{2(r-d)\Delta t} & if \quad r < d \end{cases}$$
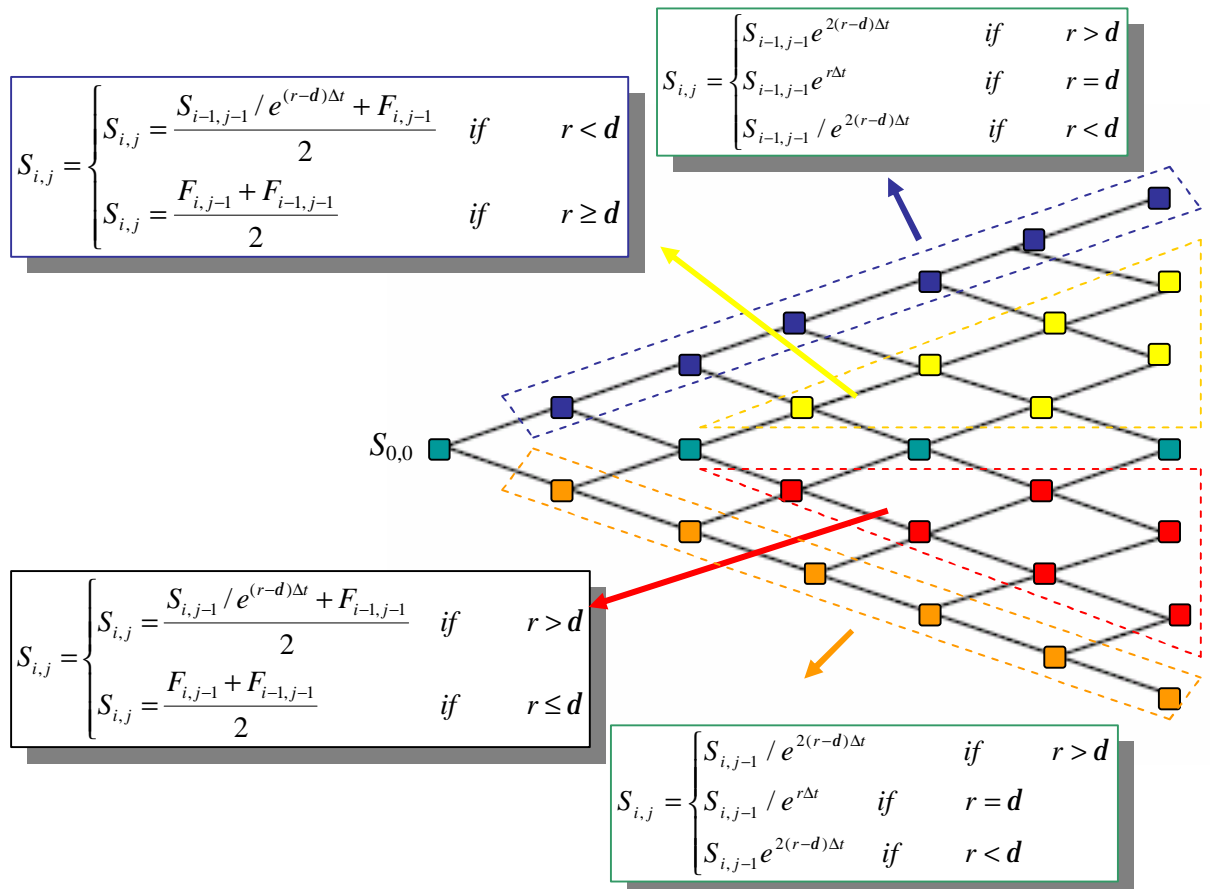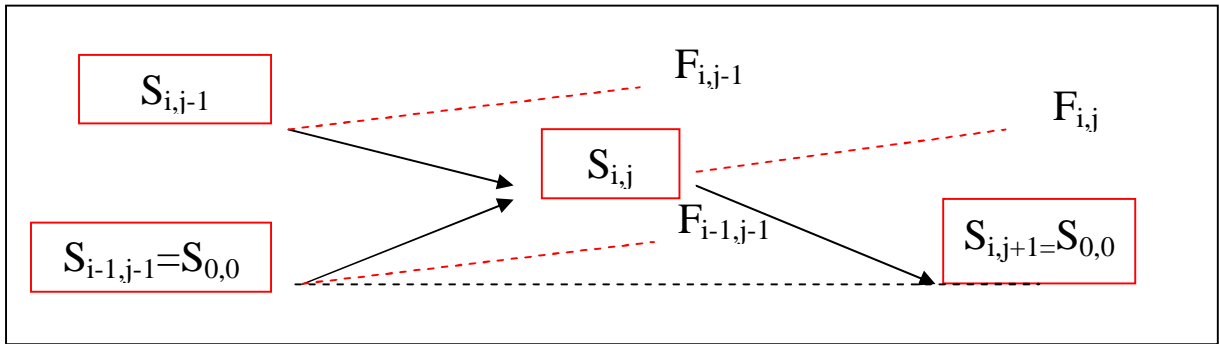
Figure 3. The no-arbitrage replacements.

Figure 4. Four nodes of the implied tree in the upper part if $r > d$.

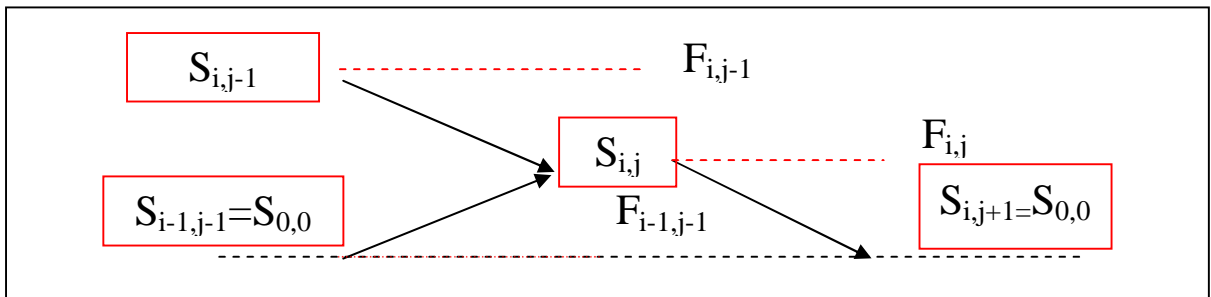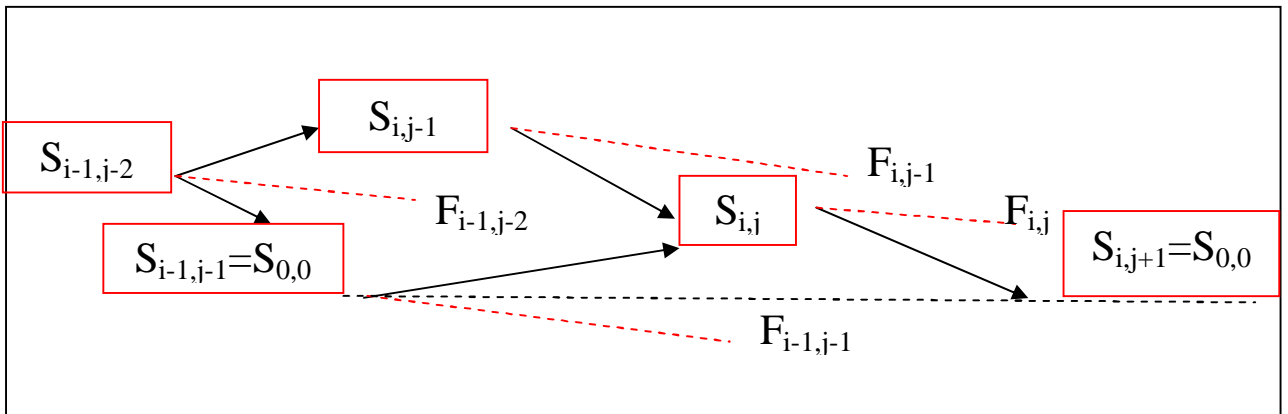Figure 5. Four nodes of the implied tree in the upper part if $r = d$.

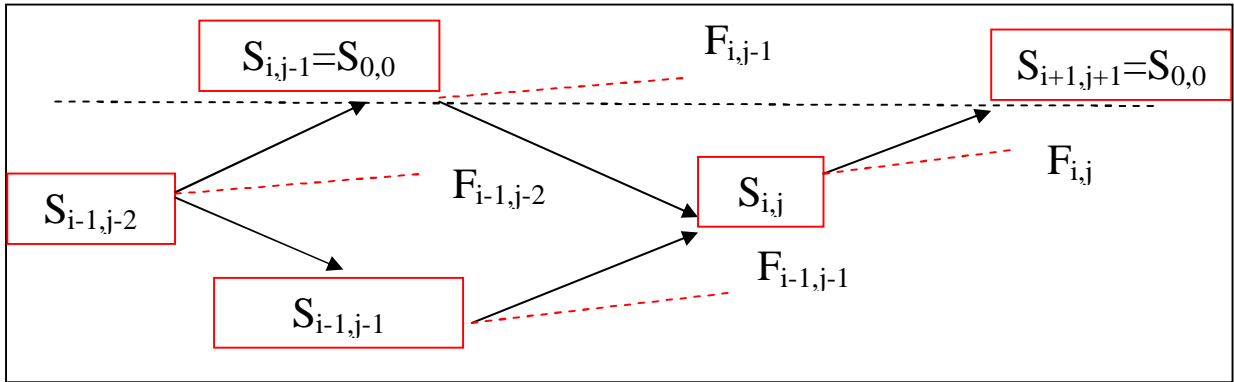Figure 6. Five nodes of the implied tree in the upper part if $r < d$.

Figure 7. Four nodes of the implied tree in the lower part if $r > d$.
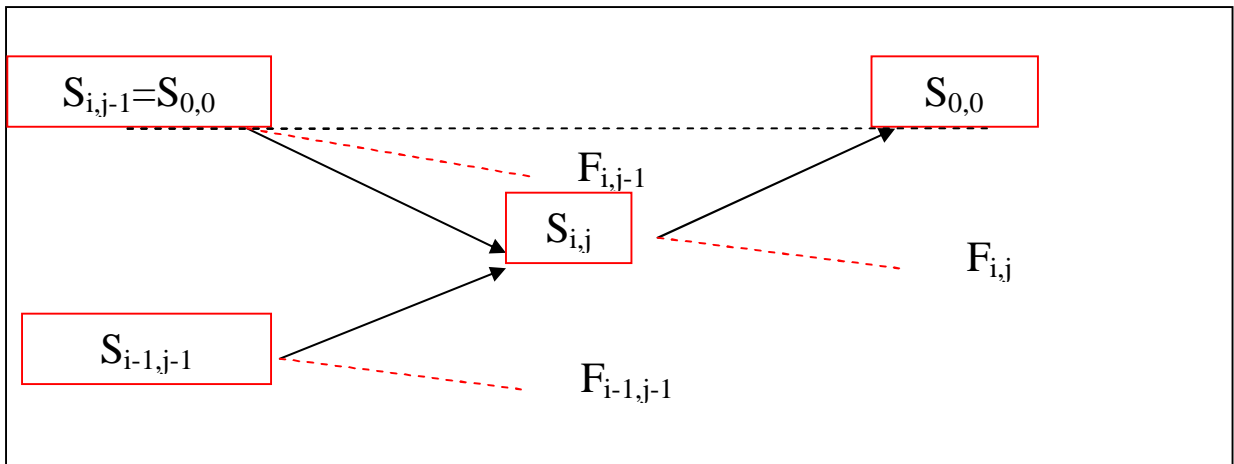

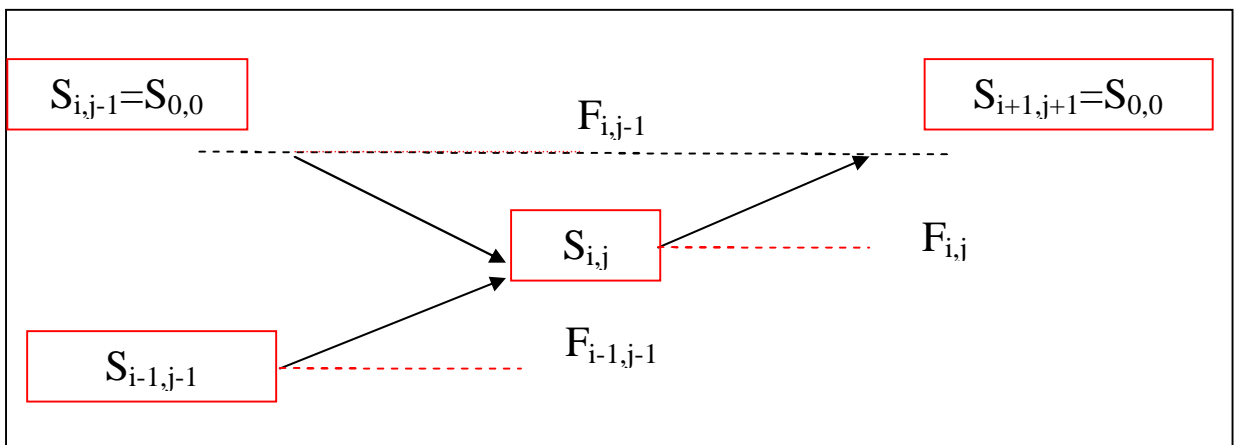Figure 8. Four nodes of the implied tree in the lower part if $r < d$.

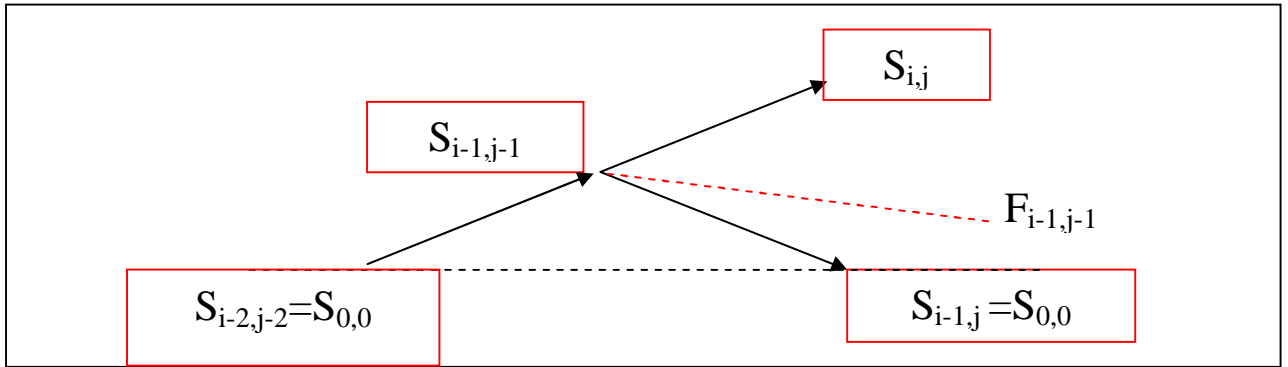Figure 9. Four nodes of the implied tree in the lower part if $r = d$.

Figure 10. Three nodes of the implied tree in the upper part at the boundary of the tree if
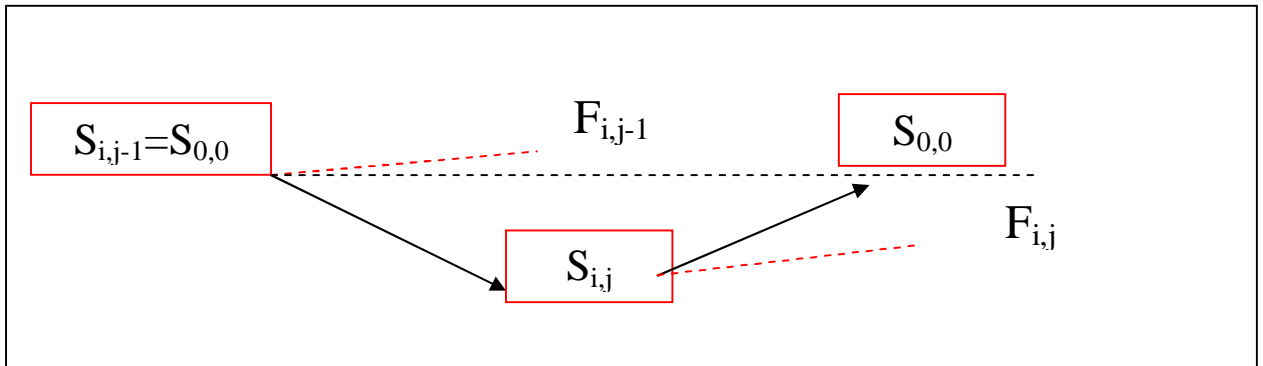
$r < d$.

Figure 11. Three nodes of the implied tree in the lower part at the boundary of the tree if

$r > d$.