CAPM-based capital budgeting and nonadditivity

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Abstract. This paper deals with the CAPM-derived capital budgeting criterion, according to which a project is profitable if the project rate of return is greater than the risk-adjusted cost of capital, where the latter depends on the systematic risk of the project. It is shown that the net disequilibrium present value implied by this criterion, widely used in corporate finance, is nonadditive. Four proofs are provided: (i) a counterexample taken from Copeland and Weston (1988), (ii) a *modus-tollens* argument showing that this notion of NPV is incompatible with additivity, (iii) a formalization showing that this NPV does not fulfil the principle of description invariance (iv) an example showing that CAPM-minded evaluators may incur arbitrage losses.

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1. Equivalent mean-variance capital budgeting criteria

This paper deals with one-period projects in a world where a security market satisfying the assumptions of the Capital Asset Pricing Model (CAPM) holds. In such a case, any asset traded in the security market lies on the Security Market Line (SML) and its rate of return is given by the relation

$$r_j = r_f + \lambda \operatorname{cov}(r_j, r_m) \tag{1a}$$

$\lambda := \dfrac{\bar{r}_m - r_f}{\sigma_m^2}$ (see other notational conventions at the end of the paper) or, equivalently,

$$\frac{\bar{\mathrm{F}}_j - V_j}{V_j} = r_f + \lambda \frac{\operatorname{cov}(\mathrm{F}_j, r_m)}{V_j}. \tag{1b}$$

Eqs. (1a)-(1b) may be restated in net-present-value terms:

$$\mathrm{NPV}_j = -V_j + \frac{\bar{\mathrm{F}}_j}{1 + r_f + \lambda \dfrac{\operatorname{cov}(\mathrm{F}_j, r_m)}{V_j}} = 0; \tag{1c}$$

solving (1c) for V_j we have the well-known certainty-equivalent form $V_j = \dfrac{\bar{\mathrm{F}}_j - \lambda \operatorname{cov}(\mathrm{F}_j, r_m)}{1 + r_f}$.

Let us suppose a project j is available to a firm and decision must be taken about undertaking it or not. Rubinstein (1973, pp. 171-172 and footnote 10) proves that if the objective is shareholder maximization, the project is worth undertaking if

$$\frac{\bar{\mathrm{F}}_j - I_j}{I_j} > r_f + \lambda \frac{\operatorname{cov}(\mathrm{F}_j, r_m)}{I_j} \tag{2a}$$

where the right-hand side is often called risk-adjusted cost of capital. Criterion (2a) is equivalent to

$$\mathrm{NPV}_j = -I_j + \frac{\overline{\mathrm{F}}_j}{1 + r_f + \lambda \dfrac{\mathrm{cov}(\mathrm{F}_j, r_m)}{I_j}} > 0. \tag{2b}$$

Note that the covariance term in (2b) depends on project cost I_j, not on the equilibrium value of the project. The resulting covariance is a *disequilibrium* covariance, so to say, not an *equilibrium* covariance. This should be intuitively obvious: the covariance is an equilibrium covariance only if the project NPV is zero (i.e. value equals cost), which means that it lies on the SML (in such a case the CAPM equilibrium relation (1) holds).

Sebnet and Thompson (1978) show the equivalence of the criteria proposed by Hamada (1969), Bierman and Hass (1973), Rubinsein (1973), Stapleton (1971), Bogue and Roll (1974). The same criterion is found in Litzenberger and Budd (1970), where they explicitly acknowledge the equivalence of Mossin's criterion, Hamada's criterion, and Tuttle and Litzenberger's (1968) criterion. Rendleman (1978) expresses again the same criterion in his eq. (1) at p. 41 and explicitly acknowledges the difference between the disequilibrium covariance term and the equilibrium covariance term. (In the Appendix of this paper the equivalence of Rubinstein's criterion and Mossin's criterion is shown).

2. Nonadditivity

Additivity in valuation is a major tenet in finance. The net present value is acceptable as a meaningful notion only if it is additive. Formally, additivity means that

$$\mathrm{NPV}_j + \mathrm{NPV}_k = \mathrm{NPV}_{j+k} \qquad \text{for all } j, k. \tag{3}$$

In words, picking *any* pair of projects *j* and *k*, the sum of their NPVs must equal the NPV of that project (*j*+*k*) obtained by summing the cash flows of the two projects. We now show that condition eq. (3) is not fulfilled if the disequilibrium NPV of eq. (2b) is used for valuation.

2.1 Counterexample

Let us consider a very simple numerical example taken from the classical textbook by Copeland and Weston (1988). At pages 414-418 the authors present two projects and calculate their risk-adjusted cost of capital. They employ $\mathrm{cov}(r_j, r_m) = \dfrac{\mathrm{cov}(\mathrm{F}_j, r_m)}{I_j}$ to compute the risk-adjusted cost of capital (see their eq. (12.30)). Tables 1 and 2 collect all the relevant data and calculations made by the authors (rates are given in percentage). As the reader may see, the costs of capital are −9.33% for project 1 and 14% for project 2. Using these costs of capital it is very simple for an investor to calculate the NPVs of the projects. We find

$$\mathrm{NPV}_1 = -100 + \frac{\frac{1}{3}(105+115+95)}{1+(-0.0933)} = 15.808 \tag{4}$$

$$\mathrm{NPV}_2 = -100 + \frac{\frac{1}{3}(107.5+100+102.5)}{1+0.14} = -9.356\,. \tag{5}$$

Let us now consider the project obtained by summing the cash flows of project 1 and project 2 and let us calculate the cost of capital and the NPV by using again the very same formulas (see Tables 3 and 4). We find a cost of capital of 2.33%. The NPV of project (1+2) is therefore

$$\mathrm{NPV}_{1+2} = -200 + \frac{\frac{1}{3}(212.5+215+197.5)}{1+0.0233} = 3.583\,. \tag{6}$$

This means that

$$\mathrm{NPV}_1 + \mathrm{NPV}_2 = 15.808 - 9.356 = 6.452 \neq 3.583 = \mathrm{NPV}_{1+2}\,.$$

In other terms, condition (3) does not hold.

It is worth noting that if we suppose that project 1's cost is 104.6 (other things equal), then we find

$$\mathrm{NPV}_1 = 10.464 \qquad \mathrm{NPV}_2 = -9.356 \qquad \mathrm{NPV}_{1+2} = -1.091 \tag{7}$$

so that

$$\mathrm{NPV}_1 + \mathrm{NPV}_2 = 1.108 \neq -1.091 = \mathrm{NPV}_{1+2}\,.$$

Additivity is not satisfied and, in addition, we have two NPVs of opposite sign, leading to different decisions about undertaking the same course of action: if the course of action is seen as the sum of two separate projects (to be both undertaken or both rejected), then the course of action is rejected; if cash flows are seen as aggregate amounts so that the gross alternative (1+2) is evaluated as a unique alternative, then the course of action is undertaken.

2.2 Modus tollens

Let us consider project *j* whose initial outflow is I_j and the final payoff is the random sum F_j. Let us assume that

(i) additivity holds

(ii) the disequilibrium NPV is used for valuation.

It is easy to show that these assumptions imply that the project's NPV can be any real number. In fact, let α be the desired NPV and choose a pair $(h^*, k^*) \in R^2$ such that

$$k^* = \left[(\alpha + I_j) \left(R_f + \frac{\lambda}{(I_j - h^*)} \right) \mathrm{cov}(\mathrm{F}_j, r_m) - \overline{\mathrm{F}}_j \right] \frac{R_f (I_j - h^*)}{\lambda \, \mathrm{cov}(\mathrm{F}_j, r_m)} .$$

where we let $R_f := 1 + r_f$. Manipulating algebraically we get to

$$\alpha = \left[-(I_j - h^*) + \frac{\overline{\mathrm{F}}_j - k^*}{R_f + \frac{\lambda}{(I_j - h^*)} \mathrm{cov}(\mathrm{F}_j, r_m)} \right] + \left[-h^* + \frac{k^*}{R_f} \right]$$

By assumption (ii), a net present value is calculated as in eq. (2b), and α may be interpreted as the sum of two projects' NPVs: the first project costs $I_j - h^*$ and pays off the random sum $\mathrm{F}_j - k^*$, the second project costs h^* and pays off the certain amount k^*. Let us call j_1 the first risky project and j_2 the second riskless project. Given that $\mathrm{F}_j = (\mathrm{F}_j - k^*) + k^*$ and $I_j = (I_j - h^*) + h^*$ we evidently have $j = j_1 + j_2$ (the two project are constituents of project j). By assumption (i) we have

$$\alpha = \mathrm{NPV}_{j_1} + \mathrm{NPV}_{j_2} = \mathrm{NPV}_j .$$

As α is any real number, then the NPV of project j is whatever number one wants it to be. To avoid this nonsense, one is bound to conclude, by modus tollens, that the two assumptions (i) and (ii) cannot simultaneously hold. In other terms, the CAPM-based notion of (disequilibrium) NPV as expressed in eq. (2) is incompatible with the notion of additivity.

2.3 Description invariance

Let us consider again project j above and let

$$f(h,k) := \left(-(I_j - h) + \frac{\overline{\mathrm{F}}_j - k}{R_f + \dfrac{\lambda \operatorname{cov}(\mathrm{F}_j, r_m)}{(I_j - h)}} \right) + \left(-h + \frac{k}{R_f} \right)$$

with $(h,k) \in \mathrm{R}^2$. Additivity implies that $f(h,k)$ is constant under changes in *h* and *k*:

$$f(h_1,k_1) = f(h_2,k_2) \qquad \text{for any} \quad h_1,k_1,h_2,k_2 \in \mathrm{R}. \tag{8}$$

To see that condition (8) does not hold, we just need to calculate the first partial derivatives of the function. After simple algebraic manipulations we find

$$\frac{\partial f(h,k)}{\partial h} = -\frac{\lambda \operatorname{cov}(\mathrm{F}_j, r_m)(\overline{\mathrm{F}}_j - k)}{[R_f(I_j - h) + \lambda \operatorname{cov}(\mathrm{F}_j, r_m)]^2}, \tag{9a}$$

and

$$\frac{\partial f(h,k)}{\partial k} = \frac{1}{R_f} - \frac{1}{R_f + \dfrac{\lambda \operatorname{cov}(\mathrm{F}_j, r_m)}{(I_j - h)}}. \tag{9b}$$

which are not identically zero. This means that the function $f(h,k)$ is not invariant with respect to *h* and *k* or, equivalently, additivity is not fulfilled. Therefore, valuation changes depending on the way a course of action is depicted (see Magni, 2002, section 4) and evaluators do not abide by the principle of description invariance, whose violations are known as "framing effects" (Tversky and Kahneman, 1981; Kahneman and Tversky, 1984; Soman, 2004).

2.4 Arbitrage loss

As seen, an NPV-minded decision-maker (DM) adopting eq. (2) may frame courses of action in different but logically equivalent ways obtaining different valuations. This implies that he is open to possible arbitrage losses, as in the following case. Suppose an arbitrageur offers an NPV-minded DM an agreement according to which they exchange the same cash flows generated by project 1, with the arbitrageur taking a short position (he will be the borrower) and the DM a long position (he will be the lender); but the arbitrageur warns the DM that if he accepts this agreement he will have to pay a 15-euros fee. The NPV of project 1 is 15.808 euros (see eq. (4)), which represents the maximum fee the DM is willing to pay in order to accept the agreement. Being 15<15.808 he accepts. Now suppose the arbitrageur offers the DM the opportunity of exchanging project (1+2)’s cash flows where the arbitrageur will be now the lender and the DM will be the borrower; but if the DM accepts he will receive a 4-euros prize. The NPV for the DM is -3.583 euros (consider eq. (6) changed in sing), but $4>3.583$ so he accepts again. Finally, the arbitrageur offers the DM the opportunity of exchanging the same cash flows generated by project 2 with the arbitrageur being the borrower and the DM acting as the lender; if he accepts, he will be rewarded by the arbitrageur with a 10-euros prize. Project 2’s NPV is -9.356 (see eq. (5)), but as $10>9.356$ he accepts again. As a result, the NPV-minded DM is trapped in an arbitrage loss of 1 euro (net cash flows for the DM are summarized in Table 5. The arbitrageur’s cash flows are the same reversed in sign).[1]

Conclusions

This paper deals with the well-established CAPM-derived capital budgeting criterion and with the widespread notion of disequilibrium Net Present Value. The latter is shown to be nonadditive. This bears interesting relations to the principle of arbitrage (see Magni, 2007) and implies that this NPV may not be used for *valuation*. And although the use of the disequilibrium NPV for *decision-making* is deductively implied by the CAPM (eq. (2b) is mathematically equivalent to eq. (2a)), its very use for decision-making is unsafe, because it leaves decision makers open to arbitrage losses.

[1] If one changes the framing and aggregates the fee/prize and the initial cash flow for each alternative, and calculates NPVs, then the arbitrage loss does not occur. This is actually a violation of description invariance (existence of arbitrage losses should not depend on the way the evaluator frames the problem).

Appendix

Mossin (1969, p. 755, left column) shows that, assuming the market is in equilibrium, an investment Z will be undertaken by a firm *l* if and only if

$$\frac{1}{1+r_f}(\overline{\mathrm{F}}_Z - R\,\mathrm{cov}(\mathrm{F}_Z, V_m)) > I_Z \tag{A.1}$$

where F_Z is the cash flow generated by the project, V_m is the end-of-period value of the security market, I_Z is the investment cost, and

$$R = \frac{\overline{\mathrm{F}}_l - (1+r_f)V_l}{\mathrm{cov}(\mathrm{F}_l, V_m)} \tag{A.2}$$

with $\overline{\mathrm{F}}_l$=free cash flow of firm *l*, V_l=market value of firm *l*. Dividing both sides of (A.1) by I_Z we have

$$\frac{\overline{\mathrm{F}}_Z}{I_Z} - R\,\mathrm{cov}(r_Z, V_m) > 1 + r_f\,. \tag{A.3}$$

As $\frac{\overline{\mathrm{F}}_Z}{I_Z} - 1 = \bar{r}_Z$, eq. (A.3) becomes

$$\bar{r}_z > r_f + R\,\mathrm{cov}(r_Z, V_m)\,. \tag{A.4}$$

Letting V_0 be the current value of the market, we have $\mathrm{cov}(r_Z, V_m) = V_0\,\mathrm{cov}(r_Z, r_m)$. Therefore, we have, using (A.2) ,

$$\bar{r}_z > r_f + \frac{\overline{\mathrm{F}}_l - (1+r_f)V_l}{\mathrm{cov}(\mathrm{F}_l, V_m)} V_0\,\mathrm{cov}(r_Z, r_m) \tag{A.5}$$

which boils down to

$$\bar{r}_z > r_f + \frac{\overline{\mathrm{F}}_l - (1+r_f)V_l}{\mathrm{cov}(\mathrm{F}_l, r_m)} \mathrm{cov}(r_Z, r_m) \tag{A.6}$$

whence

$$\bar{r}_z > r_f + \frac{\bar{r}_l - r_f}{\text{cov}(r_l, r_m)} \text{cov}(r_Z, r_m) \tag{A.7}$$

where r_l is the rate of return on firm l.[2] The term $\frac{\bar{r}_l - r_f}{\text{cov}(r_l, r_m)}$ "is the same for all companies" (Mossin, 1969, p. 755, right column), so that

$$\frac{\bar{r}_l - r_f}{\text{cov}(r_l, r_m)} = \frac{\bar{r}_m - r_f}{\sigma_m^2}.$$

As a result, eq. (A.7) becomes

$$\bar{r}_Z > r_f + \lambda \, \text{cov}(r_Z, r_m)$$

or

$$\text{NPV}_Z = -I_Z + \frac{\bar{\text{F}}_Z}{1 + r_f + \beta_Z (\bar{r}_m - r_f)} > 0$$

which coincide with eqs. (2a) and (2b) respectively, with $j=Z$.

Q.E.D.

[2] V_l and r_l refer to the value and rate of return of firm *l* prior to investment Z.

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Notational conventions used in the paper

F_j = End-of-period cash flow of asset/project j

I_j = Cost of project j

$\overline{F}_j$ = Expected end-of-period cash flow of asset/project j

r_j = Rate of return of asset or project j

$\bar{r}_j$ = Expected rate of return of asset/project j

r_f = Risk-free rate of return

r_m = Market rate of return

$\bar{r}_m$ = Expected market rate of return

σ_m^2 = Variance of market rate of return

V_j = Value of asset/project j

NPV_j = NPV of asset/project j

$\mathrm{cov}(\cdot,\cdot)$ = Covariance operator

Table 1. Copeland and Weston's example –cash flows and rates of return

		Market		Project 1 ($I_1 = 100$)		Project 2 ($I_2 = 100$)	
	Probability	r_m	r_f	F_1	r_1	F_2	r_2
State 1	0.333	26	4	105	5	107.5	7.5
State 2	0.333	14	4	115	15	100	0
State 3	0.333	20	4	95	–5	102.5	2.5

Table 2. Copeland and Weston's example –relevant statistics and values

	$\bar{r}_j$	$\mathrm{cov}(r_j, r_m)$	β_j	i_j
Project 1	5.00	−0.002	−0.833	−9.33
Project 2	3.33	0.0015	0.625	14.00
Market	20.00	0.0024	1.000	

Table 3. Project (1+2) –cash flow and rates of return			
		Project (1+2) ($I_{1+2} = 200$)	
	Probability	F_{1+2}	r_{1+2}
State 1	0.333	212.5	6.25
State 2	0.333	215	7.50
State 3	0.333	197.5	1.25

Table 4. Project (1+2) –relevant statistics and values				
	$\bar{r}_{1+2}$	$\mathrm{cov}(r_{1+2}, r_m)$	β_{1+2}	i_{1+2}
Project C	4.166	−0.00025	−0.104	2.33

Table 5. Arbitrage loss		
	Time 0	Time 1
First contract (cash flows)	–100	F_1
First contract (fee)	–15	0
Second contract (cash flows)	200	$-(F_1+F_2)$
Second contract (prize)	4	0
Third contract (cash flows)	–100	F_2
Third contract (prize)	10	0
Net Cash Flows	–1	0