Monotonicity along rays and consumer duality with nonconvex preferences

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Abstract

The goal of the paper is to prove a duality relation between the direct and indirect utility function without any reference to convexity of preferences, nor quasiconcavity of the utility function. To reach this result we allow for the pricing system to be sublinear, rather than linear. The relevance of this assumption has been illustrated both in Consumer Theory, to keep account of the presence of intermediation or of bundling, and in Mathematical Finance. The main tool is a nonlinear separation theory, which uses sublinear functionals to separate points from radiant or coradiant sets. This yields a characterization of the class of functions for which the duality can be proved, namely those whose upper level sets are evenly coradiant. Such functions are nondecreasing along each rays emanating from the origin, a very weak requirement of nonsatiation of preferences, and satisfy a further technical requirement. The conditions that we obtain are necessary and sufficient and consequently they offer the minimal assumption under which a utility function coincides with the dual of the indirect utility. We underline that this further requirement is always satisfied if u is upper semicontinuous hence, in particular, if u is continuous or differentiable.

1 Introduction

The term duality, as it is used in Economic Theory, refers roughly to the possibility of describing a theory by different and equivalent points of view, by emphasizing one aspect or another, and choosing in various ways the independent variables in terms of which the analysis should be carried out.

For instance the consumer's choice can be described by means of the utility function or the indirect utility function or still by the expenditure function and, under appropriate assumptions, the knowledge of each of these is sufficient to derive the other ones.

From a mathematical perspective, the equivalence among these concepts is determined by means of separation results and therefore the assumptions needed for the duality results are described in terms of convexity of preferences, that is quasiconcavity of the utility function. The reason that makes convexity unavoidable is that the price system is a linear functional on the space X of goods and separation by means of linear functionals characterize convex sets.

The close relation between separation results and duality is underlined by the axiomatic approach of McCabe [9], which nevertheless offers no constructive examples besides the classical one.

Indeed the linearity of a pricing system has often been questioned and the literature on nonlinear prices is very large. On the other hand, to our knowledge, the analysis of nonlinear prices have never given rise to a 'nonlinear' duality theory.

Among the many instances of nonlinear pricing systems, we refer in this paper to sublinear prices, that is to functionals $p : X \to I\!R$ which are:

i) positively homogeneous, i.e. $p(\alpha x) = \alpha p(x)$ for all $x \in X$ and all $\alpha > 0$ and

ii) subadditive, i.e. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

These features can be easily illustrated thinking of a consumer who faces no discount for buying greater quantities of a single good, but being (at least sometimes) offered a discount for buying bundles containing different goods.

Another justification for a sublinear pricing system comes from the presence of intermediation. The difference between the bid price and the ask price (bid-ask spread) entails a sublinear pricing rule in that $p(x) + p(-x) \geq 0 = p(0)$. See for instance Foley [4] or, in Capital Asset Pricing Models, Jouini e Kallal [5].

If the normalized price functional $p : X \to I\!R$ is sublinear (and continuous) then the budget constraint $B(p) = \{x \in K : p(x) \leq 1\}$, where $K \subset X$ is a closed convex cone and 1 is the (unitary) income, is a closed convex set with the origin as an interior point (with respect to the relative topology of K), but not necessarily a simplex, due to nonlinearity of p.

If we consider two goods, labelled 1 and 2, and $\bar{x}_1$ and $\bar{x}_2$ are such that $p(\bar{x}_1, 0) = p(0, \bar{x}_2) = 1$, that is $\bar{x}_1$ and, respectively, $\bar{x}_2$ are the maximum quantities of goods 1 and 2 the consumer is allowed to buy if she spends all her income in just one good, then the budget set contains bundles which lies beyond the line segment joining the vectors $(\bar{x}_1, 0)$ and $(0, \bar{x}_2)$.

In this situation the positivity of prices, as illustrated by the negative slope of the budget set, is not described by the requirement that p be nonnegative on K, but rather by its monotonicity with respect to the order induced by the cone of nonnegative goods $K \subseteq X$, which is typically the nonnegative orthant $I\!R^n_+$. This requires that $x - y \in K$ implies $p(x) \geq p(y)$. This requirement can be described in analytic terms by the positivity of the subgradients of p.

Positivity of prices is not necessary for our results and we leave for a future study the analysis of this special situation, in which positivity of p is closely related to componentwise monotonicity of u and v. In the sequel we will denote by $\mathcal{P}$ the set of all possible nonnegative price functionals, that is all continuous and sublinear functionals $p : X \to I\!R$ such that $p(x) \geq 0$ for all $x \in K$. The set $\mathcal{P}$ is a convex cone which contains the set K^+ of positive linear prices.

In an economic system in which prices are allowed to be sublinear rather than linear (or linear as a special case), the dual set becomes much larger and this allows to characterize in dual terms a corrispondingly larger class of primal objects (both sets and functions).

It has recently been developed in [18] a nonlinear separation theory in which the use of sublinear separating functionals yields a characterization of particular classes of radiant and coradiant sets of the normed space X. We recall that a subset $A \subseteq X$ is called radiant if $\alpha x \in A$ for all $x \in A$ and all $\alpha \in [0, 1]$ and that the complement of a radiant sets is called coradiant, that is a nonempty set A is coradiant when $A = X$ or when $0 \notin A$ and $\alpha x \in A$ for all $x \in A$ and all $\alpha \geq 1$.

Among the results proved in [18], the most relevant for our purposes are the ones concerning coradiant sets. The first of them states that a set A of a normed space X is closed and coradiant if and only if for every point $x \notin A$ there exists a sublinear continuous functional $p : X \to I\!R$ such that $p(x) < 1$ and $p(a) \geq 1$ for all $a \in A$. In this case the vector x receives from p a value which is strictly greater than the infimum of p over A. A second result characterizes those coradiant sets which enjoy weak separation, that is $p(x) = 1 \leq \inf_A p(a)$: a subclass of sets, called evenly coradiant and containing closed coradiant sets. This is in close analogy with the use of weak separation and evenly convex sets in 'linear' duality (see e.g. [7, 8]).

The class of utility functions that we can characterize by means of duality results are the ones whose upper level sets are evenly coradiant and this means, except for a further technical requirement which will be clarified below, any function which is increasing (we should say nondecreasing) along each rays emanating from the origin.

Thus the duality results between direct and indirect utility functions we are about to describe make no use whatsoever of convexity of preferences, nor quasiconcavity of the utility function. In place of this we assume that

$$u(\alpha x) \geq u(x) \qquad \forall x \in K, \, \forall \alpha \geq 1, \tag{1}$$

where $K \subset X$ is a convex cone in X. This requires that a proportional increase in each good of the bundle x does not decrease utility, that is a very weak assumption of nonsatiation of preferences. Functions with this property will be called radiant in the sequel, in accordance with the terminology developed in [17] and [18], or equivalently increasing along rays (i.a.r.).

We define the indirect utility function in the standard way, that is a function v defined on $\mathcal{P}$ and such that

$$v(p) = \sup\{u(x) : p(x) \leq 1\}$$

and show that u coincides with the dual $v^* = u^{**}$ of v if and only if u is increasing along rays together with another technical assumption which is implied by its differentiability or continuity or even upper semicontinuity. Moreover v is quasiconvex in $\mathcal{P}$ and decreasing along rays.

In the sequel we consider a space of goods given by a closed convex cone K in a normed vector space X. We denote by X' the topological dual of X, that is the space of linear continuous functionals from X to $I\!R$ and by

$$K^+ \equiv \{\ell \in X' : \ell(x) \geq 0, \forall x \in K\}$$

the positive polar cone of K, that is the set of functionals in X' which are nonnegative on K. This is identified with the space of positive linear prices.

If $p : X \to I\!R$ is a sublinear continuous functional, we say that a linear continuous functional ℓ from X to $I\!R$ is a subgradient of p (at the origin) if $\ell(x) \leq p(x)$ for all $x \in X$. For a function $u : X \to \overline{I\!R}$, where $\overline{I\!R} = [-\infty, +\infty]$ is the set of extended real number, we will use the symbol $[u \leq k]$ for the set $\{x \in X : u(x) \leq k\}$ with $k \in I\!R$. The sets $[u < k]$, $[u \geq k]$ and $[u > k]$ are defined similarly.

An outline of the rest of the paper is the following: Section 2 introduces evenly coradiant sets and analyzes some classes of functions which are increasing along rays. We are particularly interested in those radiant functions whose upper level sets are evenly coradiant (we call them regular). We also study other classes of radiant functions with the aim to show how slight is the restriction posed by the assumption of regularity. In Section 3 we deal with dual properties of evenly coradiant sets and of regular radiant functions. The former can be described in terms of separation or, equivalently, in terms of an appropriate polarity relation. This might be used, according to an abstract scheme (see e.g. [15] or [13]), to define the (generalized) conjugate function of u, which is precisely the indirect utility function v. We do this directly and study the duality between u and v in Section 4, which contains the main results of the paper.

2 Classes of radiant utility functions

We consider a consumer whose preferences on the space K are described by a utility function $u : K \to \overline{I\!R}$.

Such function is usually considered in the economic literature to take values into $I\!R$. However the indirect utility derived from u may naturally take values into the extended real numbers $\overline{I\!R} = I\!R \cup \{+\infty\} \cup \{-\infty\} = [-\infty, +\infty]$. Since we are interested in duality results which should guarentee that u coincides with the dual of the indirect utility function, we are lead to consider the possibility that also u takes values into $\overline{I\!R}$.

Radiant functions can be defined in terms of their level sets. It is proved in [16] and easy to verify that a function $f : X \to \overline{I\!R}$ is radiant if and only if one of the following equivalent conditions is satisfied:

i) The set $[u \leq k]$ is radiant for every $k \in I\!R$;

ii) the set $[u < k]$ is radiant for every $k \in I\!R$;

iii) the set $[u \geq k]$ is coradiant for every $k \in I\!R$;

iv) the set $[u > k]$ is coradiant for every $k \in I\!R$;

v) for every $x \in K$ the function $f_x : I\!R_+ \to \overline{I\!R}$ given by

$$f_x(\alpha) = f(\alpha x)$$

is nondreasing on $I\!R_+$.

In [16, 17, 18] some subclasses of radiant and coradiant sets are studied by means of their separation properties. In this paper we will be interested in particular in evenly coradiant sets, which are defined as follows.

Definition 2.1 *A coradiant set $S \subseteq X$ is called evenly coradiant if, it holds*

$$-x \notin T(S,x) \qquad \text{for all } x \notin S, \tag{2}$$

where $T(S,x)$ is the tangent cone to S at x and $v \in T(S,x)$ if there exist sequences $\{v_n\} \to v$ and $\{t_n\} \to 0^+$ such that $x + t_n v_n \in S$.

Evenly coradiant sets are important for their separation properties, as will be shown in Section 3. It is easy to see that a closed coradiant set is evenly coradiant, but the converse is not true in general. The origin cannot belong to an evenly coradiant set A nor to its closure. Indeed, for any set S, if $x \in \mathrm{cl}\, S$ then $0 \in T(S,x)$ and then if $0 \in \mathrm{cl}\, A$, it holds $0 \in T(A,0)$ against the definition. Moreover $T(S,x) = \emptyset$ if $x \notin \mathrm{cl}\, S$ and then condition (2) makes sense only for points $x \in \mathrm{cl}\, A \backslash A$. If A is nonempty, from a graphical point of view, the only restriction posed by (2) is that x should not stand on a cusp of A whose tangent is directed to the origin. For the results which follow we will be particularly interested in those radiant functions whose upper level sets are evenly coradiant.

Definition 2.2 *A radiant function $u : X \to \overline{I\!R}$ is called regular if the level sets $[u \geq k]$ are evenly coradiant for every $k \in I\!R$.*

As we noticed above, if the level sets $[u \geq k]$ are closed and coradiant (hence if u is radiant and upper semicontinuous, in the sequel u.s.c.) then u is regular. If we consider functions u which are not u.s.c., it is not easy to use condition (2) to see what type of restriction is imposed to a radiant function by the requirement that its upper level sets are evenly coradiant. Just by rewording condition (2) it is possible to say that a function u is regular if and only if, when $u(x) < k$ then $u(x - t_n x_n) < k$ for all sequences $\{t_n\}$ converging to 0^+ and all sequences $\{x_n\}$ converging to x. To have a better grasp to this notion, it is easier to study a subclass of regular radiant functions.

Definition 2.3 *A function $u : K \to \overline{I\!R}$ is said to be coherently radiant if for all $x \in K$ there exists $\delta > 0$ (with $\delta \leq 1$) such that:*

$$u(x - tz) \leq u(x) \qquad \forall t \in (0, \delta) \quad \forall z \in B(x, \delta), \tag{3}$$

where $B(x, \delta)$ is the closed ball of radius δ around x.

Condition (3), which implies monotonicity along rays, requires the existence of a small 'petal' with apex at x and directed toward the origin in which the function u cannot take values higher than $u(x)$. Equivalently, condition (3) exclude cases in which the strict upper level set of u at x, $[u > u(x)]$ has a tangent direction at x in the direction $-x$. This can also be described by a differential property of u at the point x. We recall that the upper Hadamard directional derivative of the function u at the point x in the direction $d \in X$ is given by

$$u_H^+(x, d) = \limsup_{\substack{d_n \to d \\ t_n \to 0^+}} \frac{u(x + t_n d_n) - u(x)}{t_n}.$$

Indeed we have the following characterization, whose proof can be easily adapted from [17, Thm. 3.6].

Proposition 2.4 *For the function $u : K \to \overline{I\!R}$ the following are equivalent:*

a) u is coherently radiant;

b) For all $x \in K$, $-x \notin T(S, x)$, where $S = [u > u(x)]$;

c) the strict level sets $[u > k]$ are evenly radiant for every $k \in I\!R$;

d) $u_H^+(x, -x) \leq 0$.

Condition (c) above can be used to show that all coherently radiant functions are regular. Indeed the class of evenly coradiant sets is closed under intersection and the equality

$$[u \geq k] = \bigcap_{s<k} [u > s],$$

which holds for all functions u, shows that all (weak) upper level sets $[u \geq k]$ are evenly coradiant if the strict upper level sets $[u > s]$ have this property.

For an example which shows that the converse relation is not generally true, we may refer to the following function $f : I\!R^2_+ \to I\!R$:

$$f(x, y) = \begin{cases} 1 & y \geq x \geq 0 \quad \text{and} \quad \sqrt{x^2 + y^2} \geq 1 \\ \sqrt{x^2 + y^2} & y \geq x \geq 0 \quad \text{and} \quad \sqrt{x^2 + y^2} < 1 \\ y/x & 0 \leq y < x \quad \text{and} \quad \sqrt{x^2 + y^2} \geq 1 \\ y\sqrt{x^2 + y^2}/x & 0 \leq y < x \quad \text{and} \quad \sqrt{x^2 + y^2} < 1 \end{cases}$$

The function f coincides, outside the unit ball, with a function which is nonnegative, positively homogeneous of degree zero, and hence constant on every rays, and continuous except at the origin. The value of f inside the unit ball increases linearly along every ray, so that f is nondecreasing and continuous along rays. Thus f is continuous on $\mathbb{R}^2_+$. Hence its upper level sets $[f \geq k]$ are closed and coradiant for every $k \in \mathbb{R}$. Since every closed coradiant set is evenly coradiant, the function f is regular. To show that f is not coherently radiant, consider any $k \in [0, 1)$ and the strict level set $[f > k]$. If we consider a point $P = (x, kx)$ outside the unit ball, the function f takes (sufficiently close to P), the value k for all points which stay on the ray defined by P and condition (3) is not satisfied since $f(P) = k$ and for every δ we can find $t \in (0, \delta)$ and $z \in B(P, \delta)$ such that $f(P - tz) > k$.

Note also that condition (d) has a much simpler form if u is differentiable at x, with gradient vector $\nabla u(x)$. In this case we have $u^+_H(x, d) = \nabla u(x) \cdot d$ and we can say that u is coherently radiant at x if and only if $\nabla u(x) \cdot x \geq 0$, which in turn is true for all differentiable radiant functions. This shows that every differentiable radiant functions are coherently radiant and hence evenly coradiant. A similar argument shows that every Lipschitz continuous radiant function is coherently radiant. Simple examples in $\mathbb{R}^2$ can be given to show that there might exists continuous radiant functions which are not coherently radiant.

3 Dual properties of radiant functions

Closed coradiant sets and evenly coradiant sets enjoy particular separation properties which make them analogous to closed convex and evenly convex sets. Such properties are studied in [18].

Proposition 3.1 *For a set $S \subseteq X$ the following are true.*

a) S is closed and coradiant if and only if for every point $x \notin S$ there exists a sublinear continuous function $p : X \to \mathbb{R}$ such that $p(x) < 1$ and $p(s) \geq 1$ for all $s \in S$.

b) S is evenly coradiant if and only if for every point $x \notin S$ there exists a sublinear continuous function $p : X \to \mathbb{R}$ such that $p(x) \leq 1$ and $p(s) > 1$ for all $s \in S$.

It is clear that every closed coradiant set is evenly coradiant. If we change in Proposition 3.1 the word sublinear with linear we obtain a well known characterization of closed convex (and, respectively, evenly convex) coradiant sets. The term evenly convex describe precisely those convex sets which can be seen as intersection of open halfspaces. We borrow the same terminology for those coradiant sets which can be seen as intersection of open level sets $[p > 1]$, where p is sublinear.

The result which is more interesting for us in Proposition 3.1 is part (b). It amounts to say that for every bundle $x \in K$ there exists a sublinear price system p such that x gives the maximum utility on the budget set $B(p) = \{x \in K : p(x) \leq 1\}$ defined by the price p. Indeed let $u(x) = k$ and $S = [u > k]$. Then $x \notin S$ (if u is strictly increasing along rays, then x lies on the boundary of S) and, by Proposition 3.1 (b), there exists a sublinear pricing system p such that $p(x) = 1$ and $p(y) > 1$ for all bundles $y \in S$. Then $B(p)$ contains no bundle whose utility is greater than $k = u(x)$.

It is possible to give a geometric illustration of Proposition 3.1 using the fact that nonnegative sublinear continuous functions are completely characterized by their level sets $[p \leq 1]$ (see e.g. [6]).

Indeed if p is sublinear, continuous, nonnegative then $S = [p \leq 1]$ is closed, convex with $0 \in \operatorname{int} S$. If conversely S is closed, convex with $0 \in \operatorname{int} S$, then the function $p(x) = \inf\{\lambda > 0 : x \in \lambda S\}$ is sublinear, continuos, nonnegative with $S = [p \leq 1]$. Thus Proposition 3.1 can be expressed in a geometric form by saying the following:

a) A set S is closed and coradiant if and only if for every point $x \notin S$ there exists an open convex set C such that $0 \in C$, $x \in C$ and $C \cap S = \emptyset$.

b) A set S is evenly coradiant if and only if for every point $x \notin S$ there exists a closed convex set C such that $0 \in \operatorname{int} C$, $x \in C$ and $C \cap S = \emptyset$.

Again part (b) is the one the fits more closely our economic setting, in that it gives a geometric interpretation for the requirement that u be regular, that is its upper level sets be evenly coradiant. Indeed if there were a displacement from $x \notin S$ which is tangent to S and leading to the origin, then there could be no sublinear pricing system giving x as the maximum of utility in the budget set $B(p)$.

The separation properties described in Proposition 3.1 can equivalently be described in terms of polarity relations between sets of X and sets of the space $\mathcal{P}$ of sublinear continuous functions. For a polarity between X and $\mathcal{P}$ we mean a map P which associates a set in $\mathcal{P}$ to every set in X and satisfies

$$P\left(\bigcup_{i \in I} S_i\right) = \bigcap_{i \in I} P(S_i).$$

We will make use of the following polarity relations between X and $\mathcal{P}$: for every set $S \subseteq X$, we define its polar $S^{\triangleright}$ and its strict polar $S^{\triangleleft}$ as

$$S^{\triangleright} = \{p \in \mathcal{P} : p(s) \geq 1, \forall s \in S\}$$

and

$$S^{\triangleleft} = \{p \in \mathcal{P} : p(s) > 1, \forall s \in S\}.$$

Repeating the same procedure we can obtain the bipolars

$$S^{\rhd\rhd} = \{x \in X : p(x) \geq 1, \forall p \in S^{\rhd}\}$$

and

$$S^{\lhd\lhd} = \{x \in X : p(x) > 1, \forall p \in S^{\lhd}\}.$$

It is easy to see that both $S^{\rhd\rhd}$ and $S^{\lhd\lhd}$ contain S. Proposition 3.1 can be used to prove that the sets $S \subseteq X$ for which $S^{\rhd\rhd} = S$ are precisely those which are closed and coradiant, while $S^{\lhd\lhd} = S$ holds if and only if S is evenly coradiant. For the use of convex polarity (the one in which linear functions are used instead of sublinear ones) in Economic Theory one may refer to [14].

Note that both the polar sets $S^{\rhd}$ and $S^{\lhd}$ are convex sets in $\mathcal{P}$. Indeed if p_1 and p_2 are elements of $S^{\rhd}$, that is if $p_i(s) \geq 1$ for all $s \in S$ and for $i = 1, 2$ and we take $t \in [0, 1]$, then $(tp_1 + (1-t)p_2)(s) = tp_1(s) + (1-t)p_2(s) \geq 1$ for all $s \in S$, whence $tp_1 + (1-t)p_2 \in S^{\rhd}$.

It will be useful in the sequel to note also that for a function u which is radiant and regular it holds $[u \geq k] = [u \geq k]^{\lhd\lhd}$, since these functions are defined precisely through the requirement that their upper level sets are evenly coradiant.

4 Indirect utility and duality

We are ready to introduce the indirect utility function derived from u, according to the usual definition. Given a utility function $u : K \to \overline{I\!R}$, the indirect utility function associated to u is the function $v : \mathcal{P} \to \overline{I\!R}$ given by

$$v(p) = \sup\{u(x) : p(x) \leq 1\}.$$

Remark 4.1 The usual domain of the indirect utility function is the polar cone K^+ of nonnegative prices. Functionals in K^+ are increasing with respect to K. Analogously we might restrict the domain of v to the set $\mathcal{P}^+$ of those sublinear pricing functionals which are increasing with respect to K, i.e.

$$\mathcal{P}^+ = \{p \in \mathcal{P} : x, y \in K,\ y - x \in K \Rightarrow p(y) \geq p(x)\}.$$

This surely brings our analysis closer to reality. For results in which the indirect utility v is defined on $\mathcal{P}^+$ (rather than $\mathcal{P}$), we should give an analogue to Proposition 3.1 in which the pricing functional is guaranteed to stay in $\mathcal{P}^+$. This is not generally true, but can be given when u is nondecreasing with respect to K. We will deal with this problem in some future research.

The definition of the indirect utility function v can be seen as a particular instance of generalized conjugate function (see [8] for the relevance of generalized conjugation theory in

Economics). Abstract conjugation can always be defined by means of a polarity relation P (see e.g. [10, 15]) and, if we decided to follow this route, it would lead to a second conjugate function which coincides with u exacly when the upper level sets of u coincide with their second polar. In this case, using the polarity $P = \triangleleft$, we might deduce some properties of v from known results about generalized conjugation theory. For the sake of clarity and completeness will will not do so.

The relation between the definition of indirect utility function and the strict polarity defined above leads us to use at times the symbol $u^{\triangleleft}$ instead of v. Such relation depends on a strong link between level sets of v and those of u. Indeed we can show that the lower level sets of v are polar to the strict upper level sets of u.

Proposition 4.2 *For every function $u : K \to \overline{I\!R}$ and its indirect function $v = u^{\triangleleft}$ it holds*

$$[v \leq k] = [u > k]^{\triangleleft}, \qquad \forall k \in I\!R.$$

Proof: It holds, for each $k \in I\!R$:

$$\begin{aligned}
p \in [v \leq k] &\iff v(p) \leq k \\
&\iff u(x) \leq k \qquad \forall x \in [p \leq 1] \\
&\iff p(x) \leq 1 \Rightarrow u(x) \leq k \\
&\iff u(x) > k \Rightarrow p(x) > 1 \\
&\iff p(x) > 1 \qquad \forall x \in [u > k] \\
&\iff p \in [u > k]^{\triangleleft}
\end{aligned}$$

□

Since a polar set is always convex, we note that the indirect utility function derived from any utility u is quasiconvex on $\mathcal{P}$. More precisely, recalling that, for every fixed $x \in K$, the function $p \mapsto p(x)$ is linear on $\mathcal{P}$, we notice that any indirect utility function v is evenly quasiconvex, that is its lower level sets are intersection of open 'halfspaces' of the form $H(x) = \{p \in \mathcal{P} : p(x) > 1\}$.

Another property which is always verified by indirect utility functions in their usual (linear) setting, is that they are nonincreasing. This follows from the requirement that they are defined on K^+. For us the following is true: every indirect utility function is nonincreasing along rays in $\mathcal{P}$. Thus the usual properties of the indirect utility function are preserved in the new setting.

Following the usual construction we can introduce the dual (conjugate) of the indirect utility, as the function $v^{\triangleleft} = u^{\triangleleft\triangleleft} : K \to \overline{I\!R}$ given by

$$v^{\triangleleft}(x) = \inf\{v(p) : p(x) \leq 1\}.$$

It is easy to show that it holds $u^{\triangleleft\triangleleft}(x) \geq u(x)$ for all $x \in K$. Indeed we have

$$\begin{aligned} u^{\triangleleft\triangleleft}(x) &= \inf\{u^{\triangleleft}(p),\, p(x) \leq 1\} \\ &= \inf_{p:\, p(x)\leq 1} \{ \sup_{z:\, p(z)\leq 1} u(z)\} \\ &\geq u(x). \end{aligned}$$

Moreover $u^{\triangleleft\triangleleft}$ is radiant. Indeed taking $\alpha \geq 1$ we have that

$$\{p : p(\alpha x) \leq 1\} \subseteq \{p : p(x) \leq 1\}$$

and hence

$$u^{\triangleleft\triangleleft}(x) = \inf\{u^{\triangleleft}(p) :\, p(x) \leq 1\} \leq \inf\{u^{\triangleleft}(p) :\, p(\alpha x) \leq 1\} = u^{\triangleleft\triangleleft}(\alpha x).$$

Our main interest is in the following question: under what conditions we might prove that $u^{\triangleleft\triangleleft} = u$? What are the minimal conditions on u which guarantee that the above equality holds?

Such questions are answered in [7] in the usual case of linear prices. For a function $f : X \to \overline{I\!R}$ we indicate with $\underline{f}$ the greatest lower semicontinuous function which minorizes f and with $\overline{f}$ the smallest upper semicontinuous function which majorizes f.

Proposition 4.3 *[7, Thm. 2.4] Let $u : K \to \overline{I\!R}$. Then there exists $v : K^+ \to \overline{I\!R}$ such that*

$$u(x) = \inf\{v(p) :\, \langle x, p\rangle \leq 1\}, \qquad \text{for all } x \in K \tag{4}$$

if and only if u is nondecreasing, evenly quasiconcave and satisfies the condition

$$u(x_0) \geq \lim_{\alpha\to 1^-} \underline{u}(\alpha x_0) \qquad \forall x_0 \in K. \tag{5}$$

Moreover v can be taken nonincreasing, evenly quasiconvex with

$$v(p_0) \leq \lim_{\alpha\to 1^-} \overline{v}(\alpha p_0) \qquad \forall p_0 \in K^+.$$

Under these conditions v is unique, namely v is the smallest function for which (4) holds; furthermore v is the indirect utility function associated with u.

If a function f is evenly quasiconvex, then its lower level sets $[f \leq k]$ can be seen as intersection of open halfspaces $[\ell < \gamma]$ with $\ell \in X'$ and $\gamma \geq 0$. Condition (5) is needed to guarentee that $\gamma > 0$ so that $p = \ell/\gamma$ can be taken as pricing functional. It is indeed quite close in spirit to condition (2) with $S = [u \geq u(x)]$. Indeed it can be shown that these conditions coincide for quasiconcave functions.

We want to show that a result similar to Proposition 4.3 is true in the present setting, i.e. the coincidence beteween u and the conjugate of its indirect utility v holds true if and only if u is radiant and regular.

Proposition 4.4 *Let $u : K \to \overline{I\!R}$. Then there exists $v : \mathcal{P} \to \overline{I\!R}$ such that*

$$u(x) = \inf\{v(p) : \langle x, p \rangle \leq 1\}, \qquad \text{for all } x \in K \tag{6}$$

if and only if u radiant and regular. Moreover v can be taken nonincreasing along rays and evenly quasiconvex. In particular the smallest function v for which (6) holds is the indirect utility function associated with u, namely

$$v(p) = \sup\{u(x) : p(x) \leq 1\}.$$

Proof: Suppose that $u(x) = \inf\{v(p) : p(x) \leq 1\}$ for some function $v : \mathcal{P} \to \overline{I\!R}$. To show that u is regular, choose $k \in I\!R$ and $x \notin [u \geq k]$ that is $u(x) < k$. Then there exists $\bar{p} \in \mathcal{P}$ with $\bar{p}(x) \leq 1$ such that $v(\bar{p}) < k$. Moreover, if $z \in [u \geq k]$ then

$$v(p) \geq u(z) \geq k \qquad \text{for all } p \text{ such that } p(z) \leq 1,$$

which is the same as

$$p(z) > 1 \qquad \text{for all } p \text{ such that } v(p) < k.$$

Hence we found $\bar{p} \in \mathcal{P}$ such that $\bar{p}(x) \leq 1$ and $\bar{p}(z) > 1$ for all $z \in [u \geq k]$ and u is regular.

To prove the converse, let u be regular and take $v(p) = \sup\{u(x) : p(x) \leq 1\}$. Then it is easy to show that $u^{\lhd\lhd}(x) \geq u(x)$ for all $x \in K$. Reasoning by contradiction, suppose that there exists some $\bar{x}$ such that $u^{\lhd\lhd}(\bar{x}) > u(\bar{x})$. Then it holds

$$u^{\lhd\lhd}(\bar{x}) > k > u(\bar{x}) \tag{7}$$

for some $k \in I\!R$. Since $\bar{x} \notin [u \geq k]$ and u is regular, then there exists $\bar{p} \in \mathcal{P}$ such that $\bar{p}(\bar{x}) \leq 1$ and $\bar{p}(z) > 1$ for all $z \in [u \geq k]$. The latter condition means that $\bar{p}(z) \leq 1$ implies $u(z) < k$ and hence

$$v(\bar{p}) = \sup\{u(x) : \bar{p}(x) \leq 1\} \leq k.$$

Recalling that $\bar{p}(\bar{x}) \leq 1$, we can deduce that

$$u^{\lhd\lhd}(\bar{x}) = \inf\{v(p) : p(x) \leq 1\} \leq v(\bar{p}) \leq k,$$

and this is a contradiction to (7). Thus $u^{\lhd\lhd}(x) = u(x)$ for all $x \in K$.

The indirect utility function is nonincreasing along rays and evenly quasiconvex.

If there exists another function $\bar{v} \in \mathcal{P}$ such that $u(x) = \inf\{\bar{v}(p) : p(x) \leq 1\}$, then, for arbitrary $p \in \mathcal{P}$ and $x \in K$ satisfying $p(x) \leq 1$, it holds $\bar{v}(p) \geq u(x)$ and hence

$$\bar{v}(p) \geq \sup\{u(x) : p(x) \leq 1\} = v(p).$$

□

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